Primary
National Strategy

Supporting children with gaps in their mathematical understanding

Wave 3 mathematics

Using the pack

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Acknowledgements

Many people have contributed to this pack. We want to acknowledge their contribution, thanking particularly the teachers, teaching assistants, mathematics consultants and other LEA staff who provided a great deal of useful feedback.

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Aims

The aims of the Primary National Strategy Wave 3 mathematics pack are:

- to develop practice in supporting children's mathematical development and self-confidence by providing a suggested model for Wave 3 mathematics intervention;
- to support the identification of specific areas of mathematics that can prevent children achieving expected levels of progress;
- to increase children's rates of progress by using targeted approaches to tackle fundamental errors and misconceptions;
- to reduce the proportion of children achieving below level 3 in mathematics by the end of Key Stage 2.

Introduction

Three 'waves'

Provision for effective mathematics learning and teaching can be described in terms of three 'waves' of intervention.

Wave 1 The effective inclusion of all children in high quality learning and teaching of mathematics in the daily mathematics lesson.

Wave 2 Additional time-limited provision in the form of small-group intervention to accelerate progress and enable children to work at age-related expectations.

Wave 3 Additional time-limited provision to enhance the progress of identified children where Waves 1 and 2 are not, on their own, having the desired effect. This will involve focused teaching activities which tackle fundamental errors and misconceptions that are preventing progress.

Introducing the Primary National Strategy Wave 3 mathematics materials

These materials have been developed during a Wave 3 mathematics pilot with 27 LEAs. Feedback has influenced the revision of the teaching materials and their presentation in the pack.

The materials:

- are aimed mainly at Key Stage 2 children, having been designed with age appropriate contexts and approaches;
- follow the principles for successful Wave 3 intervention that have been identified by research;
- aim to increase children's rate of progress by providing focused teaching activities which tackle fundamental errors and misconceptions;
- are applicable to any child who, for any reason, demonstrates fundamental errors and misconceptions;
- focus on the most commonly occurring types of mathematical difficulties with number and calculation;
- are not intended to be worked through from start to finish as a self-contained programme;
- provide a model that can be used by any adult working with a child who has demonstrated a need for a Wave 3 intervention.

The pack contains:

- Two sets of A4 booklets, one focusing on common errors/misconceptions in addition and subtraction and the second on common errors/misconceptions in multiplication and division.

  In the booklets are teaching materials referenced by year group to the National Numeracy Strategy *Framework for teaching mathematics* Key objectives.
- An A4 book, *Resources and index of games*.

  In this book are lists of mathematics equipment and everyday materials referenced in the A4 booklets, photocopiable resource sheets and an index of games contained in the A4 booklets.
- This A4 book, *Using the pack*. Within this book are:

  – management guidance (whole-school and classroom);

  – charts for tracking children's learning in addition and subtraction, and multiplication and division;

  – a professional development session to introduce Wave 3 mathematics support, with particular reference to the use of the Primary National Strategy Wave 3 mathematics materials.
- An interactive CD-ROM providing direct access from electronic versions of the tracking charts to the teaching materials in pdf and Word document formats. This enables the teaching materials to be easily adapted.

The Primary National Strategy Wave 3 mathematics materials: purpose and rationale

This section considers messages from research that indicate the need for, and structure of, effective Wave 3 provision.

The significance of effective Wave 3 provision for children with mathematical difficulties

Although in 2004 there was some improvement in the proportion of children achieving below level 3 in mathematics by the end of Key Stage 2, this proportion has not changed significantly over the last four years. Research shows that targeted interventions in mathematics can have a significant impact on children's performance and self-confidence.

A research review of what works for children with mathematical difficulties was commissioned by DfES and published in 2004[1].

Messages from the research review:

The research review suggests that mathematical difficulties:

- are common, often quite specific, and show considerable individual variations;
- are equally common in boys and girls, in contrast to language and literacy difficulties which are more common in boys;

[1] *What works for children with mathematical difficulties?* (DfES research report 554), available from DfES publications (tel:0845 60 222 60) or can be downloaded from the website at www.dfes.gov.uk/research.

- can take several forms. The causes for such difficulties are varied and include, for example, individual characteristics, inadequate or inappropriate teaching, absence from school resulting in gaps in mathematics learning, lack of preschool home experience with mathematical activities and language;

and that:

- children with mathematical difficulties typically combine significant strengths with specific weaknesses;
- some children have particular difficulties with the *language* of mathematics;
- difficulty in remembering number facts is a very common component of arithmetical difficulties, often associated with dyslexia;
- some children can remember many number facts, but seem to lack strategies (including suitable counting strategies) for working out calculations when they do not know the answer; other children are the reverse of this;
- other common areas of difficulty include word-problem solving, representation of place value and the ability to solve multi-step arithmetic problems.

The research review endorses the following points:

- Children's difficulties with calculation are highly susceptible to intervention. These interventions can take place successfully at any time and can make an impact.
- Individualised work with children who are falling behind in number and calculation can have a significant impact on their performance.
- The amount of time given to such individualised work does not, in many cases, need to be very large to be effective. Short but regular interventions of individualised work may bring a child to the point where they can profit much better from the whole-class teaching that they receive.
- It is important to find out what specific strengths and weaknesses an individual child has and to investigate particular misconceptions and incorrect strategies
- Interventions should ideally be targeted towards an individual child's particular difficulties. If they are so targeted, most children will not need very intensive interventions.

The design of the Primary National Strategy Wave 3 mathematics materials

Guiding principles

The materials evolved as feedback was provided through pilot LEAs and in response to relevant research.

The guiding principles informing the design are:

- flexibility so that teachers can adapt them;
- sharing the purpose of each activity with the child to encourage reflection on, and ownership of, learning;
- highlighting and modelling key vocabulary throughout;
- teaching activities finishing with related activities for whole-class use, where appropriate;
- use of a variety of images and models, aiming to include some the child may not have met before;

- linking mathematics to familiar and relevant contexts;
- integrating and exemplifying mathematical problem solving;
- inclusion of games among teaching activities, possibly for sharing with parents and carers.

Mathematical themes

The following are fundamental to the approach.

- Using and applying mathematics has been integrated. Often there are several opportunities for problem-solving within one activity, but in each, one particular opportunity has been highlighted. Aspects such as the following are incorporated:
  - encouraging children to discuss and explain in order to support development of their mathematical reasoning;
  - opportunities for children to make choices are woven into the activities, for example selecting numbers and devising calculations;
  - encouraging children's own recording to communicate mathematical thinking, focusing on efficiency;
  - opportunities for evaluating the efficiency of methods of calculation.
- Development is emphasised and key vocabulary is listed in each activity. It is important for adults to use correct mathematical language. To facilitate this, examples are given in words, for example, '725 × 3' is accompanied by what the adult could say to the child: 'Seven hundred and twenty-five *multiplied by* 3'.
- There is a focus on progression in counting from the earliest stages through to Year 6 to support the development of secure counting skills.
- Throughout the materials, there is emphasis on the process of estimating first, then calculating and then checking. This is denoted by the following icon:

- Decimals are addressed within meaningful contexts, for example via displays on a calculator and as a part of measure.
- Structured equipment and everyday materials are used to model mathematical concepts, supporting children's mathematical thinking and development of mental imagery. Some links to ICT resources such as the Primary National Strategy Interactive Teaching Programs (ITPs) are included.
- A wide range of resources is used in the teaching sessions. Teachers' selection of these to suit the needs of their children is an important part of adapting the materials.

Assessment overview

The materials reflect best practice in assessment for learning as a key tool for raising achievement through:

- use of questions to elicit information about children's understanding;
- sharing the purpose of the activity with the learners;
- encouraging children's reflection on their learning and identification for themselves of possible next steps.

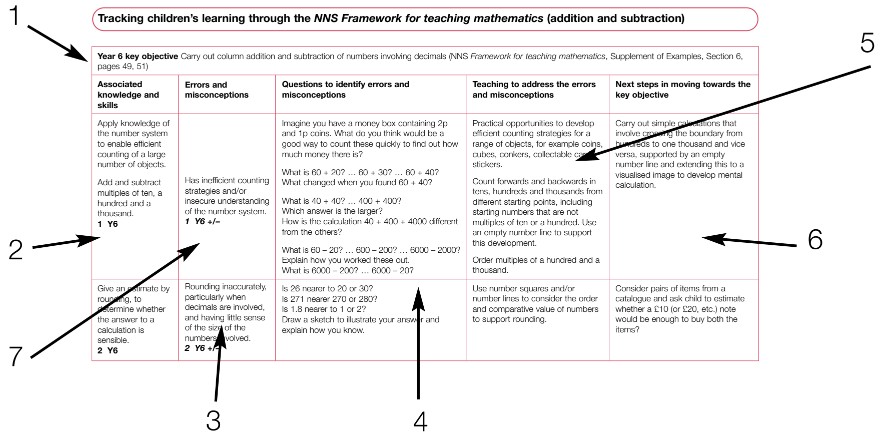
Details of the materials’ structure

The materials focus on a selection of the key objectives in National Numeracy Strategy *Framework for teaching mathematics*, namely, addition and subtraction, and multiplication and division objectives. Research shows that children’s difficulties with calculation are highly susceptible to intervention and that individualised work with children who are falling behind in number and calculation has a significant influence on their performance.

In order ***to exemplify progression in calculation, Reception, Year 2, Year 4 and Year 6 have been chosen as representative milestones***. Under each year group heading, associated knowledge and skills that contribute to understanding of the year group key objective are listed. (See first column of tracking chart.)

The whole primary age range is represented in the progression in the chart. The year group labels provide a convenient link to the National Numeracy Strategy *Framework for teaching mathematics* progression in number and calculation. As they use the chart, teachers will need to ‘track back’ to find the error or misconception appropriate for the child, ***irrespective of the year group to which it is attributed in the progression***.

Tracking charts



Tracking children’s learning through the *NNS Framework for teaching mathematics* (addition and subtraction)

Year 6 key objective Carry out column addition and subtraction of numbers involving decimals (NNS *Framework for teaching mathematics*, Supplement of Examples, Section 6, pages 49, 51)

Associated knowledge and skills	Errors and misconceptions	Questions to identify errors and misconceptions	Teaching to address the errors and misconceptions	Next steps in moving towards the key objective
Apply knowledge of the number system to enable efficient counting of a large number of objects. Add and subtract multiples of ten, a hundred and a thousand. **1 Y6**	Has inefficient counting strategies and/or insecure understanding of the number system. ***1 Y6 +/–***	Imagine you have a money box containing 2p and 1p coins. What do you think would be a good way to count these quickly to find out how much money there is? What is 60 + 20? … 60 + 30? … 60 + 40? What changed when you found 60 + 40? What is 40 + 40? … 400 + 400? Which answer is the larger? How is the calculation 40 + 400 + 4000 different from the others? What is 60 – 20? … 600 – 200? … 6000 – 2000? Explain how you worked these out. What is 6000 – 200? … 6000 – 20?	Practical opportunities to develop efficient counting strategies for a range of objects, for example coins, cubes, conkers, collectable cards, stickers. Count forwards and backwards in tens, hundreds and thousands from different starting points, including starting numbers that are not multiples of ten or a hundred. Use an empty number line to support this development. Order multiples of a hundred and a thousand.	Carry out simple calculations that involve crossing the boundary from hundreds to one thousand and vice versa, supported by an empty number line and extending this to a visualised image to develop mental calculation.
Give an estimate by rounding, to determine whether the answer to a calculation is sensible. **2 Y6**	Rounding inaccurately, particularly when decimals are involved, and having little sense of the size of the numbers involved. ***2 Y6 +/–***	Is 26 nearer to 20 or 30? Is 271 nearer 270 or 280? Is 1.8 nearer to 1 or 2? Draw a sketch to illustrate your answer and explain how you know.	Use number squares and/or number lines to consider the order and comparative value of numbers to support rounding.	Consider pairs of items from a catalogue and ask child to estimate whether a £10 (or £20, etc.) note would be enough to buy both the items?

1 Key objective.

2 This column lists associated knowledge and skills that contribute to understanding of the key objective.

3 Common errors and misconceptions linked to specific knowledge and skills are listed to support diagnosis of children’s difficulties.

4 Questions in this column can be used to help the teacher decide where the child’s difficulties lie.

5 Examples of the types of teaching activity in the A4 booklets (see below).

6 This column provides ideas to develop when the child has improved their understanding of the identified difficulty. The teacher can make use of these ideas to consolidate understanding and extend thinking.

7 Code referencing to an A4 teaching unit.

Six essential areas to support a child’s learning in calculation are the basis of the Primary National Strategy *Using models and images to support mathematics teaching and learning in Years 1 to 3* (DfES 0508-2003 GCDI) and the focus on these is reinforced in the Wave 3 mathematics pack.

These areas are:

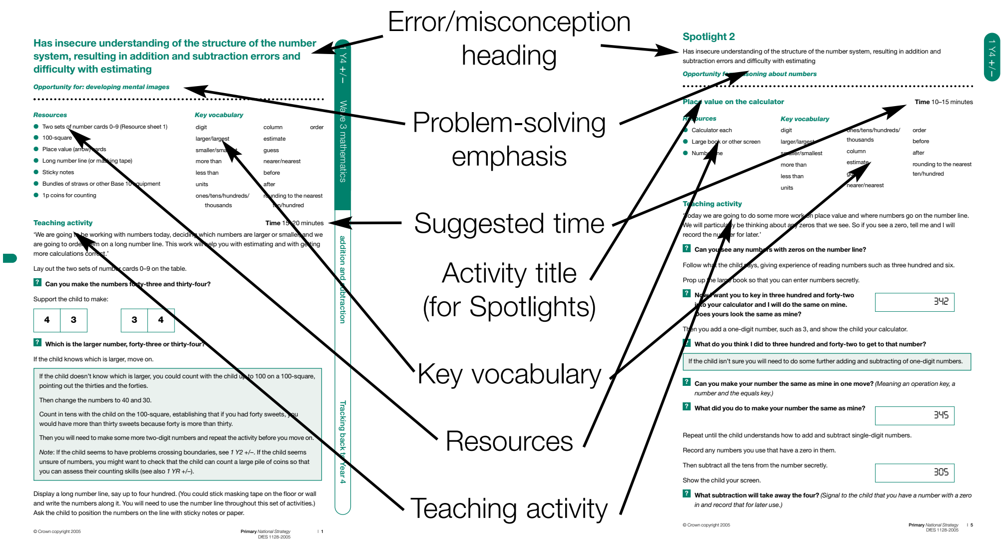
- ordering numbers;
- counting on and back;
- partitioning and recombining;
- addition and subtraction facts within 20 (not just those that total 20);
- understanding of the four operations;
- problem-solving strategies.

A4 booklets – teaching units

The structure of each booklet is as follows:

- focus error/misconception;
- opening teaching activity addressing error/misconception;
- a number of Spotlights (short focused teaching activities from which to select);
- final Spotlight, which includes assessment opportunities, often encompassed in a game, key vocabulary checklist, and intended learning outcomes list.

Opening teaching activity ***Spotlight***



Specific icons are used to improve access to the text:

Icons

Questions are incorporated for teachers to select from and add their own as appropriate.

Whole-class follow-on activity.

Symbol reminding of the necessity to estimate, calculate, then check.

This variation of the game is harder.

This variation of the game is easier.

12 x 2 = 24

Text within this symbol indicates an opportunity for recording.

Text within a shaded box indicates alternative approaches for a child who is having difficulty with the activity.

Additional game at the end of some teaching units.

Management guidance

This section outlines management issues connected with Wave 3 mathematics intervention and provides some case studies to illustrate a range of practice developed during the Primary National Strategy Wave 3 mathematics pilot.

Whole school

This section discusses issues connected with Wave 3 provision in the whole school context.

Leadership and management

There will be a number of decisions to make in connection with Wave 3 provision to suit the school's organisation. Some examples are:

- strategic planning of the coordination of Wave 3 provision;
- responsibility for coordination of Wave 3 provision;
- clarification of roles of the headteacher, deputy head, mathematics coordinator, SENCO, teachers and teaching assistants;
- continuing professional development for staff;
- processes for targeting Wave 3 provision;
- timing of Wave 3 sessions;
- storage and access to Wave 3 materials and resources;
- organisation of children to allow involvement of other children when necessary;
- opportunities for involving parents and carers;
- communication about children's progress and implications for schools' assessment procedures.

During the pilot, schools evolved a variety of solutions. Some of these are described in the case studies.

One Year 3 class teacher in the pilot occasionally replaced the mental/oral starter of her mathematics lessons with a whole-class session in the playground or hall. She used large number cards, etc. for activities based on the Wave 3 mathematics pilot materials.

Often in pilot schools, teaching assistants were given extra training: sometimes in a centrally organised LEA session, sometimes as part of a school-led activity for a group of teaching assistants, sometimes by working with teachers sharing ideas while looking at video clips from *Using models and images to support mathematics teaching and learning in Years 1 to 3*.

In one pilot school, the teacher started the day ten minutes earlier and worked with children just for ten minutes; sometimes working with one child, sometimes involving others.

During the trialling of the materials, several schools gave extra training to teaching assistants so that they could work one-to-one with the child on the Spotlights the teacher selected. (The extra training was given within the school, and sessions with the LEA numeracy consultant were also organised for teaching assistants from several schools.)

Using teaching assistants in this way was often successful, but headteachers, SENCOs, mathematics coordinators and teachers identified the vital need for the teacher to work with the child on the opening teaching activity and final Spotlight in order to assess the progress the child had made and to decide on next steps.

One of the headteachers in a pilot school said that she realised that she needed to be much more involved with what was going on in Wave 3 mathematics. She said that there would need to be some changes to roles as well as the timetable changes the school had already made to accommodate Wave 3 work. She realised that asking two teaching assistants to use the materials to support an NQT had overburdened the teaching assistants, although she was very impressed by what they had achieved.

The headteacher realised that the teacher had lost touch with the children with whom the teaching assistants were working. The headteacher decided to timetable teacher and teaching assistant planning and review sessions during whole-school assemblies each week.

A pilot school replanned its staffing budget, increasing the hours worked by some teaching assistants so they could work in partnership with class teachers on the Wave 3 intervention.

Inclusion

Understanding the importance of children retaining their entitlement to a daily mathematics lesson, schools in the pilot trialled a wide range of strategies for making time for Wave 3 provision, for example:

- during registration;
- part of lunch time;
- during afternoon teaching sessions;
- during the mental/oral starter;
- during whole-class mathematics group time.

Principles that schools adhered to were flexibility and the intention that Wave 3 provision should be for a short, focused period of time, rather than timetabled week after week for the same child.

The materials have been designed to enable teachers to continue Wave 3 activities within a whole-class context. Suitable activities are indicated at the end of many Spotlights by the icon:

Games are included in many sets of teaching activities. Some pilot schools used these beyond the children targeted for Wave 3 intervention both in school and as support for parents and carers to continue the focused teaching on particular errors/misconceptions. Some of the teaching units contain an additional game, indicated by the icon:

An index of all the games in the teaching materials is provided in the *Resources and index of games* book.

The materials are not linked to specific types of special educational need: they do not, for example, form a programme for dyslexic or dyscalculic children, or children with dyspraxia, Down’s syndrome or cerebral palsy. They are intended to have a broad application to children with difficulties in cognition and learning, however those difficulties may have arisen.

In one pilot LEA, the observation was made that children regularly taken on holiday in the first few weeks of September, often make errors relating to place value, because they miss that area of teaching each year. Some of these children might be comparatively high achievers. A pilot school pointed this out to parents, but meanwhile implemented Wave 3 intervention to help the children catch up.

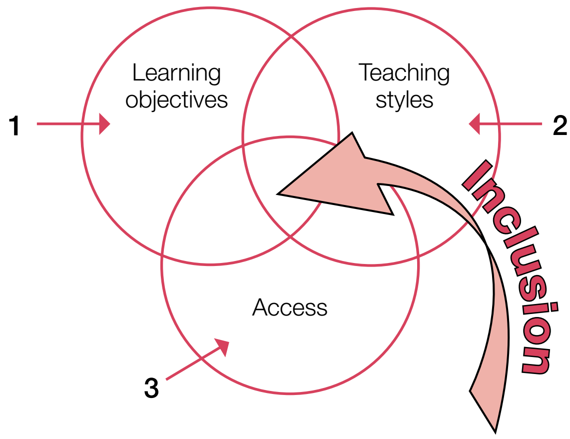
A teacher working with George, aged 9, observed his confusion with place value when working with place value (arrow) cards. She did some diagnostic assessment with him using questions from the tracking chart. Then she selected some spotlights for him to work on with the teaching assistant. The teaching assistant found George ‘resistant’ to some of the activities but she found a place value game which he played enthusiastically with another child chosen by the teacher. When the teacher came to assess George’s progress at the end of two weeks, he could work with the place value cards and said (smiling) ‘I really like maths now.’ The teacher then decided to focus on George’s mental calculation

Circles of inclusion

The different forms of support which children may need with their mathematical learning are represented on the following diagram.

These circles of inclusion provide a useful model for considering children’s needs in the context of Wave 3 mathematics provision. The teaching materials are designed to take into account the three different aspects of inclusion as conveyed in the diagram, with the primary focus on the learning objectives and teaching styles circles.

The circles of inclusion



1 Learning objectives: setting suitable learning challenges enabled by the use of the tracking children's learning charts to support the process of tracking back.

  Wherever possible, these should be linked to the class topic and focus on the earlier stages in learning of that topic. Some teachers in the pilot chose Wave 3 interventions to act as pre-learning for the targeted children before the topics became part of whole-class learning.

2 Teaching styles: implicitly addressing the needs of visual, aural and kinesthetic learners in the presentation of the teaching activities. Additional scaffolding opportunities are presented in the shaded boxes within the text.

3 Access: overcoming potential barriers to learning by using practical resources and common real-life contexts, but not attempting to cover comprehensively the range of access strategies which different children will need. The table below sets out some additional access strategies for the more common types of impairment and special educational need for which teachers and teaching assistants may need to plan.

Access strategies for mathematics

Type of impairment or difficulty	Access strategies
Dyslexia and dyscalculia	• Encouragement to use pattern and working from known to unknown to circumvent problems with rote recall of number facts • Aids to recall of basic number facts – own pocket number line, 100 square, number fact chart – to overcome problems in rote recall • Use of calculator when solving problems • Encouragement always to simplify the calculation and relate it to those they can do, so as to use estimation as a check when doing calculations • Use of kinaesthetic and visual support for learning – for example, using bead strings to teach place value and calculation • Colour-coded place value cards to help overcome problems with left-right sequencing • Jottings to note instructions and each of the steps in multi-step problems or mental calculations, to help overcome problems in working memory • Introducing new concepts using numbers the child finds easy to manipulate • Texts read aloud where necessary by a 'buddy'
Spatial and motor difficulties associated, associated, for example, with dyspraxia or cerebral palsy	• Increased use of number line when working with addition and subtraction, rather than counting objects or using fingers; own pocket number line • Teaching child to physically move objects from one side of a ruler to another, or cross them out on the page, when they must be counted • Number squares with alternate rows shaded or coloured to help them keep track of where they are • Small hole punched in top right-hand corner of numeral cards to prevent directional confusions • Use of squared paper for laying out written calculations • Mathematical symbols presented in different colours – for example, always green for +, blue for x – to prevent confusion between symbols where a difference in orientation is all that distinguishes one from another • Use of pre-prepared formats for calculations, graphs and tables • Use of appropriate software for recording calculations, graphs and tables and for drawing/manipulating shapes • Laying rulers or scales along coordinates when plotting or reading them • Teaching the child to put visual or spatial information into verbal form, and vice versa • Using non-slip matting and sticky-tack to anchor resources and paper • Recording using numeral cards or circling numerals on number lines and grids, rather than by writing
Language and communication difficulties	• Breaking down instructions and explanations into 'chunks' • Regular checks for understanding • Visual prompts and cues and the opportunity to manipulate physical resources whilst mental calculation questions are being asked • Jottings to note instructions and each of the steps in multi-step problems or mental calculations, to help overcome problems in working memory • Vocabulary charts and pocket glossaries with mathematical words and their meaning; use of mnemonics to aid recall • All words related to addition presented in one consistent colour.

> ‘We found the kinaesthetic experiences the best way to get through to them ... We used things they could touch … We did many activities in the hall on a big scale so that they were using their bodies … They loved it, especially the games … I loved seeing the smiles on their faces.’
>
> *A teaching assistant who took responsibility for resources for Wave 3 mathematics interventions in the school*

School self-evaluation of Wave 3 provision

Quantitative self-evaluation

Schools may find the following questions helpful in considering the effectiveness of their practice for low-attaining children.

- How does the percentage of our children who achieve below level 3 in mathematics at the end of Key Stage 2 or below level 2 at the end of Key Stage 1 compare with the national averages and the averages for similar schools (FSM and prior attainment)?
- How do we evaluate our children’s progress from the beginning to the end of a period of Wave 3 intervention?
- How do we use data to identify those children who could benefit from Wave 3 intervention?
- How does the progress that we achieve for children with low prior attainment compare with that achieved nationally/locally?

Qualitative self-evaluation

Schools can compare their own Wave 3 provision with a set of quality guidelines derived from research and best practice.

Key activities	Our Wave 3 provision in mathematics:	Comments/actions
Establishing priorities, analysing results and reviewing progress	• is informed by clear expectations and the tracking of individual children’s progress • involves the diagnostic assessment of children’s strengths and weaknesses • incorporates regular review and assessment of progress as an intrinsic part of the provision	
Continuing to improve the quality of learning and teaching	• is taught and overseen by personnel with appropriate skills and expertise to adapt and tailor teaching to the child’s identified needs • builds in assessment for learning as a fundamental part of the activity • ensures close connections between the intervention and the teaching of the whole class	

Leading intervention and managing and deploying resources to meet the needs of all children	• is led by members of the school's leadership team who are responsible for strategic planning • is managed by an identified member of staff who oversees the intervention on a day-to-day basis • is part of a coherent whole-school approach to the three waves of intervention • is based on as early an intervention as possible • ensures that a range of age-appropriate interventions are available • ensures that over time the entitlement of all children to a broad and balanced curriculum will be maintained • establishes regular monitoring and evaluation of the impact of Wave 3 mathematics provision	
Engaging and communicating with children and others	• ensures that children are involved in the assessment of their own learning and progress • develops children's capacity to be independent learners • develops children's self-confidence and image of themselves as successful learners of mathematics • ensures good communication and effective partnerships between all involved in children's learning, especially parents and carers	
Identifying continuing professional development needs	• uses an approach for which there is an infrastructure of support for both teachers and teaching assistants who are involved • ensures that all staff understand the whole-school approach to Wave 3 mathematics provision and their role within it	

Classroom

This section outlines features of the Wave 3 mathematics materials and their intended use in the classroom.

Key features of effective Wave 3 interventions as highlighted in the research review have been incorporated in the design of the Wave 3 mathematics pack.

In particular these are:

- *a focus on the most commonly occurring types of mathematical difficulties*

The materials focus on number and calculation, tackling areas such as understanding the structure of number and operations between numbers. Problem-solving is integrated and exemplified in the materials, and opportunities are provided for children to develop mathematical vocabulary.

- *an individualised approach based on the particular areas the child finds difficult*

  The materials reflect best practice in assessment for learning and include tracking children's learning charts that support the identification of the particular knowledge, skills and understanding with which the child needs help.

- *relatively small amounts of individualised intervention*

  The Wave 3 teaching activities provide brief, focused teaching sessions which make it possible for the child to benefit more fully from whole-class teaching. Where appropriate, the activities finish with related activities for whole-class use in order to reinforce individual learning and promote inclusive practice.

> 'All activities were seen as being short, sharp and effective. The materials were easy to use by all involved – including parents who used some as 'homework' activities.'
>
> *Extract from a report from a pilot school*
>
> 'All teachers felt that the children were enthusiastic about the activities because they were short and snappy and practical.'
>
> *Extract from a report from a pilot school*

Processes

This section provides guidance to support effective use of Primary National Strategy Wave 3 mathematics teaching materials.

Flexibility to meet identified needs

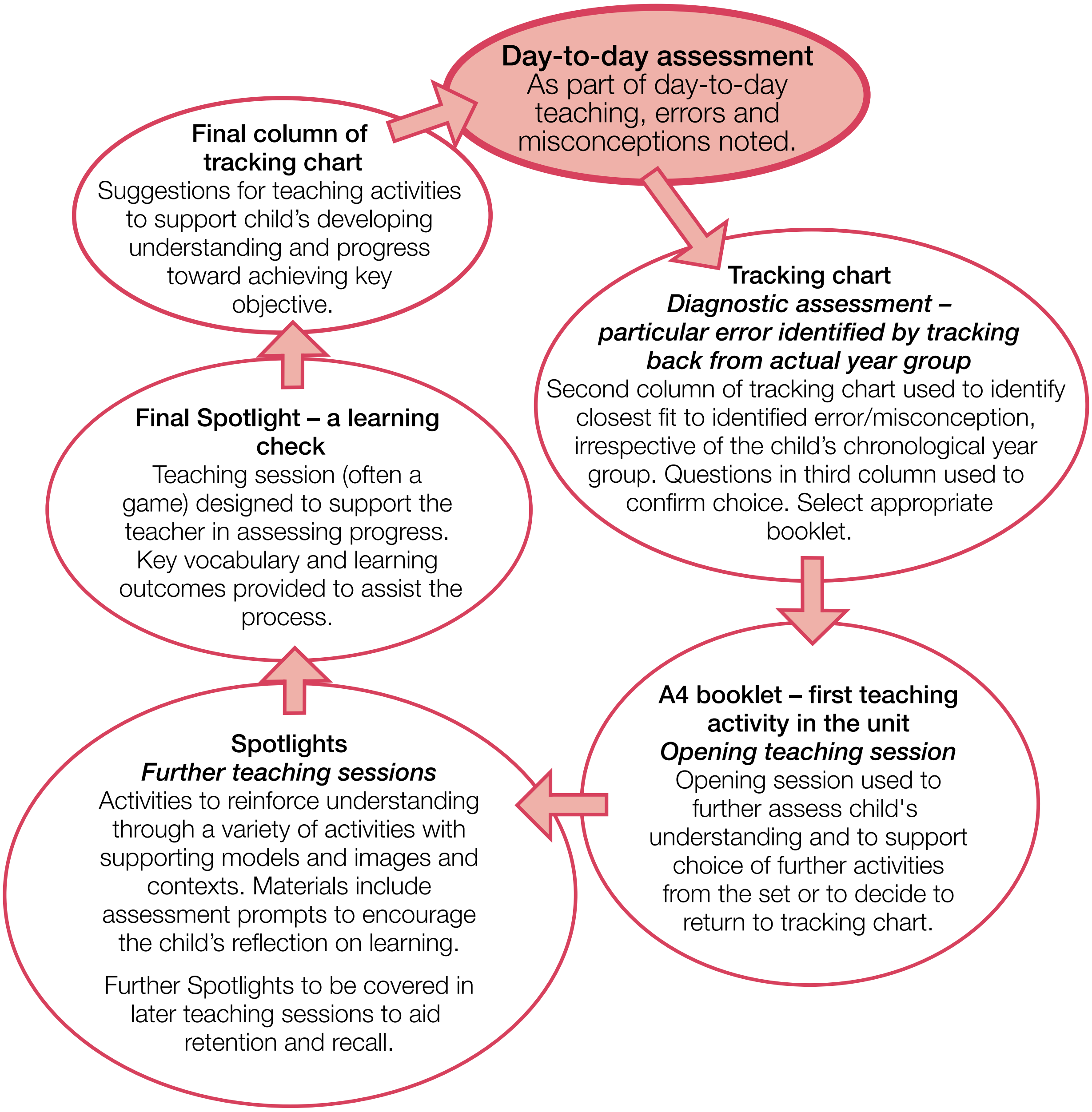
The materials are intended to be used flexibly but with the expectation that decisions about selecting from the materials will be based upon information from a tracking children's learning chart. Once a set of teaching activities related to a particular error or misconception is identified, it is up to the teacher to decide which activities within the set are relevant for a particular child.

Further flexibility is available as teachers annotate and adapt activities. Word versions of the teaching materials are provided on the CD-ROM to enable easy adaptation. As teachers become familiar with the model presented in these materials, they may well choose to use this framework to develop their own teaching materials for misconceptions and errors not represented in the pack.

Using the tracking charts and teaching materials

Day-to-day assessment is the starting point. Assessment opportunities are embedded in all the teaching materials.

The following flow chart describes the process assumed in the design of these Primary National Strategy materials.



Organisation

Particular organisational issues that pilot schools tackled in a variety of ways were:

- intervals between teaching sessions;
- resources and their management.

Intervals between teaching sessions

To aid retention and recall it is intended that teaching sessions are organised with a small interval of time between them. Pilot schools experimented with having a number of days between Spotlights but found that if the interval was too long, previous understanding was not embedded. The following case studies illustrate the strategies schools used to achieve effective timing.

During the pilot study one school tried to stick as closely as they could to the suggested pilot study timings but found that because of the demands of both timetabling and staffing, they just had to make the best ‘fit’ they could. At first their pattern for Wave 3 interventions was to work with the child on Tuesday, Wednesday and Thursday. However, they found retention so poor that they changed the whole school timetable (assemblies and so on) so that sessions could be on Monday, Wednesday and Friday. They thought that this improved retention and they were starting to send some activities home over the weekend, having recently begun to involve the parents of children involved.

Another pilot school found the suggested five minutes for the Spotlight sessions difficult to manage. They could not timetable the same children for intensive work quite as often as was suggested. They wanted to incorporate as much repetition as they could (to give the children a chance to remember important information) so class teachers targeted specific children for early morning sessions to back up the Wave 3 work that the SENCO and teaching assistants were also doing with those individuals and groups.

The timetabling varied: some groups worked in early morning sessions or after lunch. Other groups worked during the mathematics lesson: in the mental/oral starter time in small groups or as individuals, or sometimes in plenary time.

Resources and their management

Each A4 book includes lists of resources for the teaching activities.

These draw on a wide range of resources which can be categorised as:

- mathematics resources available in most schools;
- readily accessible everyday resources;
- resource sheets provided as part of the pack.

In Appendix 5, page 58, there are lists of equipment suggested for the teaching materials in the pack.

The A4 booklets include reference to resource sheets provided as part of the pack in the book, *Resources and index of games* and on the CD-ROM.

Some Interactive Teaching Programs (ITPs) are referenced within the A4 books. Others will provide a very relevant and useful resource to support children’s learning during Wave 3 sessions. A suggested list is included in Appendix 5, page 59.

ITPs can be downloaded from www.standards.dfes.gov.uk/primary for the latest versions, or from the CD-ROM.

In one pilot school, the headteacher made it a spending priority to give each teaching assistant their own pack of practical resources containing 100-squares, wipe-clean number lines, cubes, number cards, calculators, bead strings, dice, spinners, place value (arrow) cards, and so on. This funding decision followed on from a training session for teachers and teaching assistants when they watched video clips from the CD-ROM in the *Using models and images to support mathematics teaching and learning in Years 1 to 3* pack.

In another school the parents and governors provided the money for every child in the school to have a basic pack of resources in a plastic wallet.

One pilot school bought a large plastic box to store resources and this was kept centrally in school, beside the teaching materials. Resources were chosen carefully to promote a multi-sensory approach to learning; it was hoped that using colour, visual models, movement and sound would aid longer-term retention of concepts.

Frequently asked questions

? Should I only use the materials with one child at a time?

Although the materials are intended to be used with targeted children identified as having specific errors and misconceptions, the materials are flexible. In pilot schools, sometimes they were used with small groups where children had common needs.

In other year groups different children were highlighted at different times – if a misconception was identified, Wave 3 would be looked at to see if any activities would be of use or support. This meant that it was never regularly the same children – children were chosen as and when appropriate.

One teacher said she used some of the activities as a whole-class mental/oral starter when she felt things weren't going too well with a specific area. Another said she felt some activities were useful in reaffirming or refining certain skills and concepts.

A deputy head was in charge of Wave 3 mathematics interventions in her school. She kept the resources in her room and involved the mathematics coordinator and the SENCO in reviewing the materials and deciding which children would benefit from extra help.

Two children from different classes were identified for one-to-one support, and these children were withdrawn on average about twice a week, individually, after the whole-class mental mathematics start to the lesson. The child worked with the deputy head for about 20 minutes of group time, and then returned to the class for the plenary session. Sometimes, though, this proved confusing and potentially disruptive, so the child did their own plenary with the deputy head.

At other times, in some of the classes, it was decided that the deputy head would work with a group of children, using Wave 3 materials but keeping the children within the class lesson.

However do we fit in Wave 3 with everything else?

Some schools managed their Wave 3 intervention by using teaching assistant time previously deployed on non-specific in-class support for lower-attaining children in the daily mathematics lesson. Other schools used a range of strategies to give space for Wave 3 work, such as diverting some teaching assistant time from other subject areas to focus more on mathematics. In one pilot school every class did their daily mathematics lesson at the same time so that children with similar errors and misconceptions (from various year groups) could be grouped.

> To make the task easier, all year groups were asked to teach mathematics at the same time. The small groups using Wave 3 materials were going to be taught for 20 minutes, three times per week. It was agreed that the children would stay in class during the initial teacher input and would then be withdrawn and returned for the plenary. However, it became clear that returning for the plenary disrupted the remainder of the class, and did not make good use of the learning time for the group themselves. It was decided that, for the limited time of the Wave 3 intervention, the group would have their own plenary.

As the pilot progressed, many of the later teaching units included whole-class work linked to the errors and misconceptions, often with a challenge and problem-solving slant. This enabled teachers to weave Wave 3 work into the whole-class context.

I'm the mathematics coordinator; how can I get a whole-school perspective on Wave 3?

You will need to link closely with senior management, and colleagues will need to identify those children who are demonstrating misunderstandings and errors. You might need to help in the diagnostic assessment, using the tracking charts to support this.

> A pilot school mathematics coordinator said that in the new academic year she intended to teach every class in the school and work very closely with the SENCO to judge where best to focus Wave 3 interventions.

How should we select the children for Wave 3 interventions?

It will be class teachers, in their day-to-day assessment during the daily mathematics lesson and in marking written work, who will note children's errors and misconceptions. They can use the questions in the third column of the tracking charts to confirm their diagnosis.

> In one pilot school the SENCO had the main responsibility for Wave 3 mathematics because the mathematics coordinator was new. They planned from the new academic year to work more closely together.

I'm the SENCO, but my strengths are in literacy so what is my role in Wave 3 mathematics?

You will have an overview of all the Wave 3 interventions in school (not just mathematics) and you will have the SEN Code of Practice in mind. You could coordinate staff who are involved, making sure that there is good communication throughout the school, and supporting class teachers in tracking children's progress so that the impact of your Wave 3 provision can be evaluated.

Will children be withdrawn from lessons or will Wave 3 interventions sometimes or always be kept within the class lesson?

Pilot schools said their practice varied considerably. In one school, when the teacher wanted to do the Wave 3 work, she stayed in the class working with the focus group while the teaching assistant supervised the rest of the class as they worked on the task the teacher had given them.

Schools where children were regularly withdrawn identified a problem with keeping the teacher informed of those children's progress; the children were also becoming out of touch with the whole-class work.

> The Spotlights were chosen to link in with the week's plan for the class. If this was not possible the teaching assistant selected a Spotlight from an area of specific concern or repeated something done previously. The programme was not used as a scheme or in a certain order, but rather as a resource to support what was already happening in the main lessons.

> Pupils were taken out of the mental/oral starter approximately three times a week. They worked in a separate room with the teaching assistant. They worked from the Spotlights, although the sessions invariably ran over the five minutes suggested in the pilot in order to ensure that the children gained useful support.

We want to go and visit some schools where their Wave 3 work seems to be going more smoothly than ours. We thought we could send different teachers with a teaching assistant for a few visits then we will all compare notes and see if we can find a better way to organise ourselves.

SENCO

The children have definitely benefited, and their basic numeracy skills and confidence have improved.

Class teacher

Teaching assistants' expertise and opinion has been valued and this has had a positive effect on their motivation.

SENCO

We have been extremely pleased with the implementation of Wave 3 mathematics at our school. It has boosted the confidence of all of the children involved and the positive effects have been witnessed throughout the school. Many teachers have commented on the change in children's attitudes. A Year 2 teacher observed a child who was previously petrified of mathematics lessons and is now much more cooperative and happier during lessons; she is very pleased with the positive influence the group work has had.

Headteacher

Continuing professional development

This section contains suggestion for a staff meeting to introduce the Primary National Strategy Wave 3 mathematics pack and strategies for Wave 3 mathematics provision.

Outline for a staff meeting

Objectives

- To provide some background to the development of the Primary National Strategy Wave 3 mathematics materials
- To develop practice in supporting children's mathematical development and self-confidence by providing a suggested model for Wave 3 mathematics intervention
- To familiarise staff with specific detail about using the Wave 3 mathematics pack
- To consider the identification of specific areas of mathematics that can prevent children achieving expected levels of progress

Key messages

- It is important that the Wave 3 mathematics materials are seen as a suggested model for providing Wave 3 intervention, which teachers should adapt to suit the needs of their children.
- There are key features in the design of the materials to support good practice in working with children with mathematical difficulties.
- Good day-to-day assessment is fundamental in supporting effective use of Wave 3 mathematics intervention.

Resources

- A complete Primary National Strategy Wave 3 mathematics pack *Supporting children with gaps in their mathermatical understanding*
- A copy of the addition and subtraction tracking children's learning chart for each member of staff
- Copies of handouts 1, 2 and 3 from Appendix 3 for each member of staff
- Computer, etc. to demonstrate CD-ROM from pack

Introduction

1 Share the objectives for the session, setting these in the context of the three 'waves' of inclusion.

Three 'waves'

Provision for effective mathematics learning and teaching can be described in terms of three 'waves' of intervention.

Wave 1 The effective inclusion of all children in high quality learning and teaching of mathematics in the daily mathematics lesson.

Wave 2 Additional time-limited provision in the form of small-group intervention to accelerate progress and enable children to work at age-related expectations.

Wave 3 Additional time-limited provision to enhance the progress of identified children where Waves 1 and 2 are not, on their own, having the desired effect. This will involve focused teaching activities which tackle fundamental errors and misconceptions that are preventing progress.

2 Briefly mention the development of the materials, their piloting and revision in response to feedback.

3 Refer to the research background that informed the design, mentioning that key features of effective Wave 3 interventions highlighted in the research review have been incorporated in the design of the Wave 3 mathematics pack.

In particular these are:

- *a focus on the most commonly occurring types of mathematical difficulties*

  The materials focus on number and calculation, tackling areas such as understanding the structure of number and operations between numbers. Problem solving is integrated and exemplified in the materials, and opportunities are provided for children to develop mathematical vocabulary.

- *an individualised approach based on the particular areas the child finds difficult*

  The materials reflect best practice in assessment for learning and include tracking children's learning charts that support the identification of the particular knowledge, skills and understanding with which the child needs help.

- *relatively small amounts of individualised intervention*

  The Wave 3 teaching activities provide brief, focused teaching sessions which make it possible for the child to benefit more fully from whole-class teaching. Where appropriate, the activities finish with related activities for whole-class use in order to reinforce individual learning and promote inclusive practice.

Activity 1: Getting to know the Primary National Strategy Wave 3 mathematics materials

Introduce the components of the pack to illustrate what the pack has to offer. Briefly mention each of the items (see notes on page 7).

Activity 2: Understanding the purpose of the tracking children's learning charts

1 Ask colleagues to scan **Handout 1**, *Structure of Wave 3 mathematics materials*, to begin to get an overview of the tracking chart. They should then scan the addition and subtraction chart, working in pairs to identify a familiar error/misconception from the second column.

2 Ask colleagues to look at the third column of the chart to find questions to confirm their diagnosis.

How would they see themselves using these questions? One to one? Working with a small focus group including the target child?

Ask them to suggest further questions to confirm diagnosis of the chosen error/misconception.

3 Explain that the fourth column contains an outline of work they could cover when they choose a booklet and the fifth column shows how teachers could extend the work after the child has completed activities in the booklet.

4 Ask colleagues to focus on the code of their error/misconception, find the related A4 booklet and read through the activities, identifying:

- helpful aspects for teaching and learning.
- any suggestions for improving the work in the booklet to suit the needs of their children.

Take feedback, emphasising the intention that teachers modify the materials as necessary for their children.

Demonstrate the tracking back process using the CD-ROM interactive tracking chart.

Activity 3: Focusing on the teaching activities

As they look at the booklet they have selected and use **Handout 2**, *Mathematical themes and assessment approaches in the Wave 3 materials*, ask colleagues to find examples of the mathematical themes listed on the handout.

Some questions to consider:

- What are the implications for your practice?
- What guidance would you need to give to your teaching assistant to support them in working with a child, or a couple of children, with Spotlight 1 in the booklet?

At the end of Handout 2 is a list of the ways in which best practice in assessment for learning is reflected in the materials. Ask colleagues to look at this list and the booklet they have been using to identify the ways in which assessment for learning is supported.

Activity 4: Assessment and planning process

Ask colleagues, in pairs, to use **Handout 3**, *Assessment and planning process*, to draw together the elements of the suggested model for using the materials to support children's learning.

Ask for questions and suggestions about how the school can incorporate the use of this pack to support children's learning.

Activity 5: Developing Wave 3 mathematics in our school

This part of the session will need to focus on the practicalities of Wave 3 mathematics planning in the individual school.

Refer to material from this booklet to focus the discussion – See the section entitled 'Classroom' page 17.

From this meeting, the leadership team will have gathered knowledge to inform their further planning. Support will be found in the management guidance, starting on page 12.

End by referring to some of the next steps identified by the pilot schools as they planned how to further improve Wave 3 provision for their children.

I think we might need to rethink our assessment records so that there is more space for the richness of the observations we can do in Wave 3.

We're going to do more TA training.

My advice to anyone setting out to do Wave 3 would be to keep asking yourself 'Am I doing this the best way?' We kept changing what we did in the light of circumstances and we're now on our third way of doing it … The results have been fantastic – more confidence and raised self-esteem and kids smiling and wanting to do maths.

Keep doing it! It's great!

We want to involve the parents more.

Anything that gets the kids this excited about maths has to be worth putting lots of time and effort into. We just want to do more of it next year.

Appendix 1

Tracking children's learning chart – addition and subtraction

The tracking charts act as an initial diagnostic tool and a background framework for the sets of teaching materials referenced to common misconceptions and errors.

Teaching unit codes are referenced against the errors and misconceptions listed in the chart, for example *1 Y6 +/–*

Tracking children’s learning through the *NNS Framework for teaching mathematics* (addition and subtraction)

Year 6 key objective Carry out column addition and subtraction of numbers involving decimals (NNS *Framework for teaching mathematics*, Supplement of Examples, Section 6, pages 49, 51)

Associated knowledge and skills	Errors and misconceptions	Questions to identify errors and misconceptions	Teaching to address the errors and misconceptions	Next steps in moving towards the key objective
Apply knowledge of the number system to enable efficient counting of a large number of objects. Add and subtract multiples of ten, a hundred and a thousand. **1 Y6**	Has inefficient counting strategies and/or insecure understanding of the number system. ***1 Y6 +/–***	Imagine you have a money box containing 2p and 1p coins. What do you think would be a good way to count these quickly to find out how much money there is? What is 60 + 20? … 60 + 30? … 60 + 40? What changed when you found 60 + 40? What is 40 + 40? … 400 + 400? Which answer is the larger? How is the calculation 40 + 400 + 4000 different from the others? What is 60 – 20? … 600 – 200? … 6000 – 2000? Explain how you worked these out. What is 6000 – 200? … 6000 – 20?	Practical opportunities to develop efficient counting strategies for a range of objects, for example coins, cubes, conkers, collectable cards, stickers. Count forwards and backwards in tens, hundreds and thousands from different starting points, including starting numbers that are not multiples of ten or a hundred. Use an empty number line to support this development. Order multiples of a hundred and a thousand.	Carry out simple calculations that involve crossing the boundary from hundreds to one thousand and vice versa, supported by an empty number line and extending this to a visualised image to develop mental calculation.
Give an estimate by rounding, to determine whether the answer to a calculation is sensible. **2 Y6**	Rounding inaccurately, particularly when decimals are involved, and having little sense of the size of the numbers involved. ***2 Y6 +/–***	Is 26 nearer to 20 or 30? Is 271 nearer 270 or 280? Is 1.8 nearer to 1 or 2? Draw a sketch to illustrate your answer and explain how you know.	Use number squares and/or number lines to consider the order and comparative value of numbers to support rounding.	Consider pairs of items from a catalogue and ask child to estimate whether a £10 (or £20, etc.) note would be enough to buy both the items?

Tracking children's learning through the *NNS Framework for teaching mathematics* (addition and subtraction)

Associated knowledge skills	Errors and and misconceptions	Questions to identify errors and misconceptions	Teaching to address the errors and misconceptions	Next steps in moving towards the key objective
Partition whole numbers and decimal numbers, and add and subtract the constituent parts and recombine to complete the calculations. **3 Y6**	Has difficulty in partitioning numbers with zero place holders and/or numbers less than one, for example partitioning 0.45 as 0.4 and 0.05. ***3 Y6 +/–***	What is the value of the digit 1 in 3010? … 201? … 6.1? What do you notice about the value of the digit 2 in 4.2? … 0.2? … 0.25? Write the following statements: 507 is equal to 50 + 7 7403 is equal to 7000 + 40 + 3 0.75 is equal to 0.7 + 0.05 Which are true? How do you know?	Use place value (arrow) cards to partition and recombine numbers, including numbers with zero place holders and decimals, for example 10.85 and 18.05.	Provide or ask the child to select a calculation using numbers previously partitioned and ask them to explain how they can tackle it supported by place value (arrow) cards to demonstrate. Repeat with child choosing further calculations and numbers.
Add and subtract a pair of numbers that involves crossing boundaries, recognising when to adjust, to compensate and to carry numbers across the boundaries. **4a Y6** **4b Y6**	Has difficulty in choosing suitable methods for calculations that cross boundaries: addition ***4a Y6 +/–*** Has difficulty in choosing suitable methods for calculations that cross boundaries: subtraction ***4b Y6 +/–***	What would you add to 37 to make the nearest multiple of 10? What would you add to/subtract from 240 to make the nearest multiple of 100? Explain to me, using an empty number line, how you find 300 – 237. Using an empty number line, explain how you would work out 5016 + 3700. What is 217 – 6? … 217 – 7? What happens when you subtract 8 from 217? Explain how you subtract 18 from 217.	Continue to practise counting up using an empty number line and record the steps, focusing on the significance of the multiples of ten and hundred as milestones to support developing understanding of crossing boundaries. Use place value (arrow) cards to partition before adding and subtracting numbers, to illustrate the significance of columns to support calculation.	Transfer calculation practice to a vertical number line, taking the opportunity to relate this to measures. Increase the number of boundaries to be crossed, asking the child to compare calculation methods considering their efficiency.

Tracking children's learning through the *NNS Framework for teaching mathematics* (addition and subtraction)

Year 4 key objective Carry out column addition and subtraction of two integers less than 1000 and column addition of more than two such integers (NNS *Framework for teaching mathematics*, Supplement of Examples, Section 6, pages 48, 50)

Associated knowledge and skills	Errors and misconceptions	Questions to identify errors and misconceptions	Teaching to address the errors and misconceptions	Next steps in moving towards the key objective
Know the value of each digit in a three-digit number, for example that the 3 in 437 has a value of 30. Recognise the significance of each digit in numbers up to 1000 and find an estimate of the addition or subtraction calculation. **1 Y4**	Has insecure understanding of the structure of the number system, resulting in addition and subtraction errors and difficulty with estimating. ***1 Y4 +/–***	Is 43 smaller or larger than 34? What about 343 and 334? How did you decide? Which is the largest/smallest number: 216, 612 or 162? How do you know? What number would you add to 437 to make 477? How did you decide?	Use a range of place value models and images such as bundles of straws, digit cards and structured apparatus to partition numbers and read them correctly. Use a number line and place value (arrow) cards to identify position of a variety of numbers in relation to their nearest ten, nearest hundred. Make two-digit and three-digit numbers using digit cards. Change one digit at a time, for example 134 to 144, establishing the size of the change, in this case ten more. Establish whether the change is significant, for example that 154 is closer to 200 than 100, and use this to inform estimates for addition and subtraction calculations.	From a set of digit cards, ask the child to select three, of which two must be the same, for example 3, 3 and 6. Ask the child to make and read aloud different three-digit numbers, for example three hundred and thirty-six. Repeat activity. Ask the child to estimate answers for some given calculations involving two- and three-digit numbers and discuss with child the estimation methods used.
Partition two- and three-digit numbers in a number of ways. **2 Y4**	Has difficulty in partitioning, for example, 208 into 190 and 18 and 31 into 20 and 11. ***2 Y4 +/–***	What number does 20 + 10 + 1 represent? Is it the same as 20 + 11? What should I add to 190 to make 208? What other ways can you partition 208?	Use place value (arrow) cards to partition and combine numbers with zero as place holder, discussing the position of the zero as a place holder. Use number squares or a string of beads to model the partitioning of numbers in different ways, recording the associated number statements, such as 43 = 40 + 3 = 30 + 13 = 20 + 23	Use partitioning to carry out a range of two-digit calculations, discussing the way partitioning makes the calculation more manageable. Repeat with three-digit numbers.

Tracking children's learning through the *NNS Framework for teaching mathematics* (addition and subtraction)

Associated knowledge skills	**Errors and and misconceptions**	**Questions to identify errors and misconceptions**	**Teaching to address the errors and misconceptions**	**Next steps in moving towards the key objective**
Know when it is most appropriate to use column addition or subtraction rather than carrying out the calculation by another method. **3 Y4**	Does not make sensible decisions about when to use calculations laid out in columns. ***3 Y4 +/–***	What is 700 – 1? … 700 – 10? … 700 – 9? What about 30 + 20? … 30 + 21? … 30 + 27? … 296 – 57? … 328 + 187? How did you do these?	Ask the child to choose pairs of numbers to add together, highlighting where mental methods are most appropriate and ones where written methods are more appropriate.	Ask the child to identify a collection of pairs of numbers that would definitely be best tackled by written calculation methods and explain why.
Recognise that the operation of column addition can apply to more than two numbers. **4 Y4**	Has difficulty with adding three numbers in a column, except by adding the first two and then the last one. ***4 Y4 +/–***	Referring to a written addition column calculation involving three numbers, ask the child to talk through their steps in calculation.	Use place value (arrow) cards to demonstrate column addition of more than two numbers and focus on completing the calculation one column at a time.	Extend to adding more than three numbers using a column method.

Tracking children's learning through the *NNS Framework for teaching mathematics* (addition and subtraction)

Year 2 key objective Understand that subtraction is the inverse of addition; state the subtraction corresponding to a given addition and vice versa (NNS *Framework for teaching mathematics*, Supplement of Examples, Section 5, pages 25, 29, 35)

Associated knowledge and skills	Errors and misconceptions	Questions to identify errors and misconceptions	Teaching to address the errors and misconceptions	Next steps in moving towards the key objective
Count on and back in ones and tens. **1 Y2**	Makes mistakes when counting using teen numbers and/or crossing boundaries. ***1 Y2 +/–***	Choose a teen number and count backwards from your number in ones. What is the next number, the number before? Count forwards in tens from 7. What number is ten more than 27? … ten less than 97?	Use a number line and a 100-square to count forwards and backwards in ones and tens. Encourage similar counting activities using a visualised number line.	Use known facts about adding and subtracting ten to work out how to calculate using nine and eleven.
Identify pairs of numbers that add to twenty and use known number facts to add mentally. **2 Y2**	Has difficulty in remembering number pairs totalling between ten and twenty, resulting in calculation errors. ***2 Y2 +/–***	What is 3 + 5? What is 13 + 5? How did you work that out? Which pair of numbers adds to 18? Are there any other pairs?	Use rulers and number squares to support practice with recall of number pairs beyond ten and emphasise patterns in number pairs.	Extend patterns to numbers beyond 20; use 4 + 3 = 7 to deduce other number facts such as 14 + 3 = 4 + 13 and so on.
Find a difference by counting up from the smaller to the larger number. **3 Y2**	Counts up unreliably; still counting the smaller number to get one too many in the answer. ***3 Y2 +/–***	How many do I add on to get from three to eight? I've got three sherbet dips and I want eight. How many do I need to buy?	Use a number line to demonstrate counting up by jumping in ones and highlighting the equivalent larger jump. This will illustrate that the starting number position is not included in the jump.	Ask the child to choose a new jump size, for example twos or fives, to find the difference between two given numbers. How can you check your answer?

Tracking children's learning through the *NNS Framework for teaching mathematics* (addition and subtraction)

Associated knowledge skills	Errors and and misconceptions	Questions to identify errors and misconceptions	Teaching to address the errors and misconceptions	Next steps in moving towards the key objective
Recognise subtraction as taking away, finding the difference and complementary addition. **4 Y2**	Does not relate finding a difference and complementary addition to the operation of subtraction. ***4 Y2 +/–***	What is the difference between 21 and 18? How did you work it out? What operation did you use? What other ways can you think of for subtracting 18 from 21?	Illustrate the connections between different aspects of subtraction using a range of models and images, such as number lines, money, counters or Interactive Teaching Programmes (ITPs) such as *Counting on and back, Differences, Number facts*.	Provide the following three questions: What is the difference between 21 and 18? 21 – 18 = ? 18 + ☐ = 21? What do you notice about these and their answers? Make up some of your own and explain the patterns you are using.
Recognise, for example, that subtracting 13 'undoes' adding 13 and vice versa, and that this means that since 4 + 13 = 17, we can state the inverse that 17 – 13 = 4. **5 Y2**	Is insecure in making links between addition and subtraction and/or recognising inverses. ***5 Y2 +/–***	What is the answer to 30 add 20? If 30 add 20 is 50, what is 50 subtract 20? What is 17 subtract 8? Write a number sentence for this calculation. Use the three numbers to write an addition fact.	Use digit cards and number lines to make and check number statements that involve inverse pairs such as 11 + 3 = 14, 14 – 3 = 11.	Give the child statements to sort into pairs involving inverses, then into groups that make up a set of the four equivalent statements such as 24 + 3 = 27, 3 + 24 = 27 27 – 3 = 24, 27 – 24 = 3 Invite the child to make sets of equivalent statements using numbers they have chosen.
Develop and recognise patterns to help deduce other addition and subtraction facts. **6 Y2**	Does not readily use number patterns to support calculating, for example: 46 – 5 = 41, so 46 – 15 = 31, 46 – 25 = 21, etc. ***6 Y2 +/–***	What is 14 + 5? …14 + 15? …14+25? Using this information, tell me what 24 + 25 is. What is 6 – 4? …16 – 4? … 26 – 4? Now what do you think the answer is to 56 – 4? How did you work that out?	Use number squares to demonstrate and highlight patterns such as 5 + 1 = 6, 15 + 1 = 16, 25 + 1 = 26 32 – 2 = 30, 32 – 12 = 20 getting children to predict and check other cases.	Extend to include patterns where the calculations cross a boundary such as 14 + 9 = 23, 14 + 19 = 33 56 – 9 = 47, 56 – 19 = 37.

Tracking children's learning through the *NNS Framework for teaching mathematics* (addition and subtraction)

Reception key objective Begin to relate addition to combining two groups of objects, and subtraction to 'taking away' (NNS *Framework for teaching mathematics*, Supplement of Examples, Section 4, pages 14, 15, 16)

Associated knowledge and skills	**Errors and misconceptions**	**Questions to identify errors and misconceptions**	**Teaching to address the errors and misconceptions**	**Next steps in moving towards the key objective**
Count along and back on a number track to and from a given position. Count objects set out in different arrangements; begin to recognise small numbers without counting and that the number of objects is not affected by their position. Count objects that are out of reach. **1 YR**	Can only begin counting at one; inaccurately counts objects when rearranged; has no consistent recognition of small numbers of objects; lacks systematic approaches. ***1 YR +/–***	Ask the child to choose a starting number. What are the next two numbers? What number comes after the third number? What comes before your starting number? The contents panel says there are twelve. Let's check. Tip them out and put them back. How many are there now? Can you tell me how many sweets there are here without counting them? How many spots are in this picture? Throw a small number of objects onto the table. Can you count them without touching them? How did you do it? How do you know you're correct?	Children walk forwards and backwards along a large number track, along a number line, on a snakes and ladders board, counting aloud. Start from different positions; use digit cards or dice to select start and to move one forwards, two backwards, etc. Using sets of mixed objects to count and rearrange, ask children to estimate and check after each rearrangement. Put small numbers of objects in familiar and unfamiliar patterns and compare with known patterns such as spots on dice, displays on wall, etc. Count sounds such as drum beats or coins dropping into a money box; provide counters or pencils to record marks as they count; count counters or their marks on a sheet; compare ways to systematically count particular arrangements, for example window panes, squares on a grid, chairs around a table.	Cover up selected numbers for children to identify as they count; move forward and backwards from different starting points; count aloud and silently to determine position after a move. Arrange a known number of objects into two or more groups to establish that the total remains the same; count in twos; count objects arranged in pairs; use recognisable patterns of small numbers, such as 3 and 4, to introduce counting in threes and fours. Estimate the number of objects that can be counted reliably; check by counting, first touching objects then without touching.

Tracking children's learning through the *NNS Framework for teaching mathematics* (addition and subtraction)

Associated knowledge skills	**Errors and and misconceptions**	**Questions to identify errors and misconceptions**	**Teaching to address the errors and misconceptions**	**Next steps in moving towards the key objective**
Find one more and one less than a given number. **2 YR**	Misunderstands meaning of 'one more' and 'one less'; does not consistently identify the number before or after a given number. ***2 YR +/–***	Here are four counters. How many will you have if I give you one more? There are six spots showing on my dice. Imagine there is one less spot. How many spots would there be? What is one more than seven? … one less than seven?	The child counts aloud from a given starting number, stopping at particular numbers. What number comes next? What number comes before? Relate counting to a set of objects, add one more and ask children to identify how many, similarly one less; arrange objects alongside a number track and keep taking one away to link one less to the number before and similarly one more to the next number.	What is one less than six? …one more than four? Ask me another pair of these questions with the same answer. Count every other number; identify odd and even numbers; count in twos; add and take away two objects and find two more and two less than a given number; build on pattern recognition to introduce three more, three less, etc.
Say how many there are altogether by counting all the objects when combining groups for addition. Separate a given number of objects into two or more groups and say how many there are in each group. **3 YR**	Does not relate the combining of groups of objects to addition and/or does not interpret the counting of all of the objects as an answer to the question 'How many are there altogether?' ***3 YR +/–***	How many spots are there on this blue card? How many spots are there on this red card? How many spots are there altogether? There are three spoons in this cup and two spoons in this cup. How many spoons altogether? How do you know? Listen to these claps. How many were there? Now tell me how many extra claps I make. How many claps is that altogether? There are seven grapes in my lunch box. How many red grapes and green grapes could there be? Use these cubes to show how you did it.	Use containers, such as an egg box, that hold a given number of objects. Give the child groups of objects to count, for example four objects and two objects, and place the objects in the container for children to count all the objects. Hide the container and ask how many objects it holds. Recount, check and confirm. Repeat with other number pairs. Arrange a small number of biscuits on two plates. Ask the child to count the biscuits on each plate and say how many there are altogether. Move biscuits between the plates and repeat activity.	Arrange a small number of biscuits on two plates so that one plate has two more biscuits than the other. Ask the child to say how many biscuits on each plate and how many altogether. Repeat with different numbers of biscuits as appropriate. Extend to three plates.

Tracking children's learning through the *NNS Framework for teaching mathematics* (addition and subtraction)

Associated knowledge skills	Errors and and misconceptions	Questions to identify errors and misconceptions	Teaching to address the errors and misconceptions	Next steps in moving towards the key objective
Say how many are left when some objects are taken away, by counting how many objects are left. **4 YR**	Is not confident about when to stop counting when taking away (subtracting) in answer to the question 'How many are left?' ***4 YR +/–***	There are six in this box. I take away two and put them here. How many are left? Here are eight cups. Three cups are full and the rest are empty. How many are empty? We ate four of the six cakes we made. How many are left? How did you work this out?	Use a range of objects and resources such as beads on a string. Invite the child to take one away and count the rest. How many are left? Repeat the activity, inviting the child to take away different numbers and describe what has happened. Emphasise that it is the number of objects left, not the objects that we take away, that is important.	Continue to ask questions that involve finding how many are left after taking away or subtracting; ask the child to find their own way of recording their answers and to describe their solutions using a range of vocabulary such as: 6 take away 4 is 2 6 subtract 4 is 2 Start with a number and take a number away. How many are left? Ask the child to start with another number and take some away to leave the same number as before. Record the calculations before repeating the activity. Ask the child to comment on what they notice and explain how they chose their numbers.

Appendix 2

Tracking children's learning chart – multiplication and division

The tracking charts act as an initial diagnostic tool and a background framework for the sets of teaching materials referenced to common misconceptions and errors.

Teaching unit codes are referenced against the errors and misconceptions listed in the chart, for example *1 Y6 ×/÷*

Tracking children's learning through the *NNS Framework for teaching mathematics* (multiplication and division)

Year 6 key objective Carry out long multiplication of a three-digit by a two-digit integer and short multiplication and division of whole numbers (NNS *Framework for teaching mathematics*, Supplement of Examples, Section 6, pages 67 and 69)

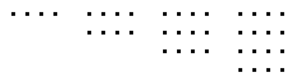
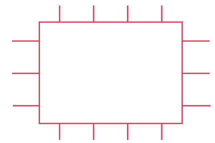
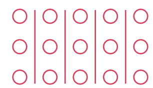
Associated knowledge and skills	Errors and misconceptions	Questions to identify errors and misconceptions	Teaching to address the errors and misconceptions	Next steps in moving towards the key objective
Know by heart all multiplication facts up to 10 × 10. Derive division facts corresponding to multiplication facts up to 10 × 10.	Refer to Year 4 chart (1 Y4 and 2 Y4) where this is described in connection with a limited range of multiplication and division facts. At Year 6 level, the number range will extend to include multiplication facts up to 10 × 10 and related division facts.			
Multiply and divide one-, two- and three-digit numbers by ten and one hundred. Apply the associative law (but not by name) to multiply up to three-digit numbers by multiples of ten and hundred, for example: 147 × 20 = 147 × 2 × 10 **1 Y6**	Misuses half-understood rules about multiplying and dividing by powers of ten and the associative law, for example: 145 × 30 = 145 000 ***1 Y6 ×/÷***	Can you tell me a quick way of multiplying a number by one thousand? I have thirty-seven on my calculator. What single multiplication should I key in to change it to three thousand seven hundred? What is 3 ÷ 1? …30 ÷ 10? …300 ÷ 100? Tell me about this pattern. If I had four thousand eight hundred on my calculator, what single division could I key in to change the display to forty-eight? 47 × 10 = 470 What do you think 47 × 20 equals?	Multiplication by ten, one hundred… using a calculator and recording numbers before and after multiplication in HTU columns. Repeat for division. Ask the child to describe patterns observed and suggest a generalisation. Ask the child to illustrate with further examples. Demonstrate that 42 × 12 is the same as 42 × 6 × 2. What other ways can we multiply forty-two by twelve? Repeat with other numbers. Use factor 'tree' to support this, for example: 12 → 4 × 3 12 → 6 × 2 12 → 3 × 4 12 → 2 × 6	What's the best way you can think of to find: 71 × 20? 202 × 6? 82 × 300?

Tracking children's learning through the *NNS Framework for teaching mathematics* (multiplication and division)

Associated knowledge skills	Errors and and misconceptions	Questions to identify errors and misconceptions	Teaching to address the errors and misconceptions	Next steps in moving towards the key objective
Present the remainder in a quotient as a whole number or fraction. **2 Y6**	Has difficulty, when appropriate, interpreting a remainder as a fraction, for example: $16 \div 3 = 5\frac{1}{3}$ ***2 Y6* ×/÷**	There are twenty-six apples for four horses, so they can have six and a half each. How do I know this?	What is a sensible answer for each of the following division calculations? 1. Divide twenty-eight eggs by six. 2. Divide twenty-seven bars of chocolate into four. 3. Divide £47 by four. 4. Divide sixty-two children into teams of ten.	Write a division question for your partner which you know will have a fraction as part of the answer. How do you know it will have a fraction as part of the answer?
Dividing by numbers smaller than one, for example: $12 \div \frac{1}{2} = 24$ $6 \div \frac{1}{3} = 18$. **3 Y6**	Interprets division as sharing but not as grouping (repeated subtraction) so is unable to interpret a calculation such as $12 \div \frac{1}{2}$. ***3 Y6* ×/÷**	How many half tomatoes can you get from three whole tomatoes? How many quarters of pizza can you get from four pizzas? Explain how to work out $5 \times \frac{1}{2}$, $5 \div \frac{1}{2}$.	Model with practical contexts and equipment, calculations such as: How many halves in six? …quarters in eight? Demonstrate associated recording. Use a variety of images and contexts to illustrate, for example, jumps on a number line, chocolate bars, etc.	When I divide my mystery number by half the answer is ten. What's my mystery number? Make up some more questions with mystery numbers.
Judge whether the answer to a multiplication or division calculation is reasonable. **4 Y6**	Is not confident in making reasonable estimates for multiplication or division calculations. ***4 Y6* ×/÷**	$600 \div 30 = 2$ Can this calculation be correct? How do you know? Try some more, such as: $540 \div 20 = 52$ $24 \times 20 = 4800$ $15 \div \frac{1}{2} = 30$ $24 \div \frac{1}{4} = 22$.	Use examples of calculations to model making estimates, for example: We know thirty-five divided by seven equals five. Is thirty-three divided by seven more or less than this? How can we decide? Repeat using calculations such as: $44 \div 10$ $207 \div 20$ Ask the child to justify the estimates they suggest.	If the question is fifty-four divided by seven, what number facts could you use to help you estimate an answer? Explain how each of the facts you suggest would be useful to help you make an estimate. What about the question five hundred and four divided by six? Choose some more questions to use.

Tracking children's learning through the *NNS Framework for teaching mathematics* (multiplication and division)

Year 4 key objective Use informal pencil and paper methods to support, record or explain multiplications and divisions, develop and refine written methods for TU × U, TU ÷ U, and find remainders after division (NNS *Framework for teaching mathematics*, Supplement of Examples, section 6, pages 56, 66, 68)

Associated knowledge and skills	Errors and misconceptions	Questions to identify errors and misconceptions	Teaching to address the errors and misconceptions	Next steps in moving towards the key objective
Know multiplication facts for 2, 3, 4, 5 and 10 times tables. **1 Y4**	Is not confident in recalling multiplication facts. ***1 Y4* ×/÷**	What is three multiplied by four? Draw me a diagram to show what this means. How do you know the answer to calculations such as three multiplied by zero and ten multiplied by zero? What is four multiplied by one? How do you know?	Provide models and images to illustrate the multiplication facts, for example multiple groups of beads on a bead string, jumps along a number line. Use relevant rows from a multiplication square as a resource on which to focus questions about the multiplication facts connected to particular numbers, illustrated with a growing array as a connected image, for example: 4 | 8 | 12 | 16 	 Cover a criss-cross pattern with a rectangle of card or paper. Ask the child to imagine the crossovers (knots, perhaps). How many lines across? How many crossovers (knots) in each? What multiplication fact will tell you how many crossovers are under the paper? Ask the child to make up hidden crossover patterns of their own and say the accompanying multiplication facts.
Know the division facts which correspond with multiplication facts listed above. Understand the inverse relationship connecting multiplication and division. **2 Y4**	Is muddled about the correspondence between multiplication and division facts, recording, for example: 3 × 5 = 15 so 5 ÷ 15 = 3 ***2 Y4* ×/÷**	Tell me the multiplication facts this four by five array shows (support this question with a diagram of the array). And the division facts? If you know three multiplied by five equals fifteen, what number sentences using division do you also know? Draw a picture for three multiplied by six. What other multiplication and division facts can you find from the picture?	Show the child a three row by five column array, made from counters laid out. Demonstrate the multiplication fact three multiplied by five equals fifteen from the pattern:  3 × 5 = 15 How could I explain this pattern as a division fact? Show me from the counter pattern how you know your division fact?	Working with the child, set out thirty-five counters in a five by seven array. Ask the child to explain how the array can be seen to represent two multiplication and two division calculations. Ask the child to list the calculations.

Tracking children's learning through the *NNS Framework for teaching mathematics* (multiplication and division)

<table>
<tr><th>Associated knowledge skills</th><th>Errors and and misconceptions</th><th>Questions to identify errors and misconceptions</th><th>Teaching to address the errors and misconceptions</th><th>Next steps in moving towards the key objective</th></tr>
<tr>
<td>Understand the effect of multiplying whole numbers less than a thousand by ten.
3 Y4</td>
<td>Describes the operation of multiplying by ten as 'adding a nought'.
3 Y4 ×/÷</td>
<td>What is the answer to forty-six multiplied by ten? …three hundred and fifty-one multiplied by ten?
How do you know?</td>
<td>Multiply by ten using Base 10 apparatus on a ThHTU board. For example, put forty-six on the board (four tens, six ones) and label with digit cards. Multiply each piece by ten, move the pieces into the correct positions to become four hundreds, six tens, zero units and label the resulting number with digit cards. Repeat for other two-digit numbers.

Use place value (arrow) cards, multiplying the number on each place value card by ten. Replace each card by the multiplied by ten version. Bring the cards together to make the completed multiplication.</td>
<td>Extend this activity to work with three-digit numbers multiplied by ten, and then two-digit numbers multiplied by ten then multiplied by ten again.

Encourage children to record the calculations and see the pattern as follows:
46 × 10 = 460
460 × 10 = 4600
4600 × 10 = 46000</td>
</tr>
<tr>
<td>Can apply the distributive law (but not by name) to multiplying, using partitioning and recombining, for example:
14 × 3 = (10 × 3) + (4 × 3)= 30 + 12 = 42,
so
$\begin{array}{c|c|c} \times & 10 & 4 \\ \hline 3 & 30 & 12 \end{array}$

Know how to record TU × U multiplication calculated by a partitioning method in a grid format.
4 Y4</td>
<td>Does not apply partitioning and recombining when multiplying,
for example:
14 × 3 is calculated as (10 × 3) + 4 = 34,
or
$\begin{array}{c|c|c} \times & 1 & 4 \\ \hline 3 & 3 & 12 \end{array}$

14 × 3 = 312, confusing the value of two-digit numbers.
4 Y4 ×/÷</td>
<td>What number would you partition to work out twenty-seven multiplied by three?
How would you recombine your calculations?
Can you spot the mistake in this calculation:
15 × 7
= (10 × 7) + 5
= 75?</td>
<td>Roll a dice three times to generate a range of multiplication calculations, for example 3, 5, 6.

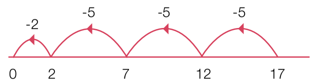
Record 36 × 5 to demonstrate and explain partitioning and recombining.

Ask child to make another different multiplication calculation to demonstrate and explain. Use place value (arrow) cards to demonstrate partitioning of two-digit numbers prior to multiplication and again to demonstrate recombining.</td>
<td>Extend to using partially completed calculations for child to complete, for example:
16 × 3 = (10 ×) + (× 3)
= 30+
= 48

$\begin{array}{c|c|c} \times & 10 & \\ \hline 3 & & 18 \end{array}$

Encourage the child to make a partially completed multiplication calculation for a partner to solve.</td>
</tr>
</table>

Tracking children's learning through the *NNS Framework for teaching mathematics* (multiplication and division)

Associated knowledge skills	Errors and and misconceptions	Questions to identify errors and misconceptions	Teaching to address the errors and misconceptions	Next steps in moving towards the key objective
Recognise that the commutative law holds for multiplication but not for division. **5 Y4**	Assumes that the commutative law holds for division also, for example, assuming that 15 ÷ 3 = 5, so 3 ÷ 15 = 5. ***5 Y4* ×/÷**	How do you work out your answer to three divided by fifteen? Take some cubes, or use a diagram to explain what this calculation means	Use objects and a number line to work out calculations such as 16÷4, reading this as divide sixteen into groups of four. How many groups are there? Then compare this to 4÷16 reading this as divide four into groups of sixteen. How many groups are there? Highlight that this is not the same.	Provide a range of division calculations to be sorted into those with answers that are whole numbers and those involving fractions, for example, 10 ÷ 5 = 5 ÷ 10 = 16 ÷ 8 = 8 ÷ 16 =
Understand the idea of remainder, and when to round up or down after division. **6a Y4** **6b Y4** **6c Y4**	Writes a remainder that is larger than the divisor, for example, 36 ÷ 7 = 4 remainder 8. ***6a Y4* ×/÷** Discards the remainder; does not understand its significance. ***6b Y4* ×/÷** Does not recognise when a remainder is significant in the decision about whether to round up or down. ***6c Y4* ×/÷**	With 29p to spend, you want to buy as many sweets at 3p each as you can afford? How many sweets can you buy? Show your working out on a number line. Which of these calculations is correct? 36 ÷ 7 = 4 remainder 8 36 ÷ 7 = 3 remainder 15 36 ÷ 7 = 5 remainder 1 36 ÷ 7 = 5. Which do you think is the best one? Why? There are thirty-two children and each tent takes three children. What is the least number of tents they will need?	Use a number line to highlight the idea of a remainder. Demonstrate repeated subtraction by 'stepping back', for example, 17 ÷ 5 can be represented by:  Provide opportunities to consider how results of calculations may be rounded up or down in order to make sense of the context. For example: • How many tables are needed to seat a class of thirty-two children if five children can sit at a table? This would involve rounding up to seat all the children. • How many £5 books can be bought with £32? This would involve rounding down to ensure enough money to pay for the purchase.	Extend to the context of money, weight and length to demonstrate how the remainder can be subdivided into smaller units. For example, £60 shared equally between eight. The remainder of £4 can be divided further to give an answer of £7.50 each.

Tracking children's learning through the *NNS Framework for teaching mathematics* (multiplication and division)

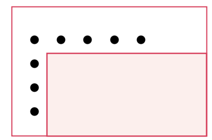
Associated knowledge skills	Errors and and misconceptions	Questions to identify errors and misconceptions	Teaching to address the errors and misconceptions	Next steps in moving towards the key objective
Know how to record division as repeated subtraction, with appropriate use of chunking. **7 Y4**	Continues to subtract twos when calculating twenty divided by two without using knowledge that two multiplied by five equals ten. ***7 Y4* ×/÷**	Ask the child to look back at a division calculation they have just completed. Can you think of a quicker way to find out an answer for this question?	Use a bead string to illustrate: • How thirty divided by two can be calculated as thirty with ten subtracted three times. Every ten is five twos. Altogether fifteen twos have been subtracted so thirty divided by two equals fifteen. • How to work out seventy-six divided by five by starting from seventy-six beads, remove ten lots of five, that is fifty, and then five lots of five, that is twenty-five. Altogether how many groups of five have been slid along the bead string? 76 -50 10 × 5 26 -25 5 × 5 1 so 76 ÷ 5 = 15 remainder 1	Ask the child to work out fifty-three divided by four using knowledge of the four times table, and to record this as repeated subtraction using not more than three steps.

Tracking children's learning through the *NNS Framework for teaching mathematics* (multiplication and division)

Year 2 key objective Understand the operation of multiplication as repeated addition or as describing an array, and begin to understand division as grouping or sharing; know by heart facts for the 2 and 10 multiplication tables; and know and use halving as the inverse of doubling. (NNS *Framework for teaching mathematics*, Supplement of Examples, section 5, pages 47, 49, 53, 57)

Associated knowledge and skills	Errors and misconceptions	Questions to identify errors and misconceptions	Teaching to address the errors and misconceptions	Next steps in moving towards the key objective
Carry out repeated addition, recognise the relationship between multiplication and repeated addition, and use the associated vocabulary of multiplication. **1 Y2**	Still counts in ones to find how many there are in a collection of equal groups; does not understand vocabulary, for example, 'groups of', 'multiplied by'. ***1 Y2* ×/÷**	Show an array of six rows of two. How many sets of two can you see? How many are there altogether? Repeat with six rows of ten.	Reveal an array of twos, one row at a time. Encourage the child to imagine the next row and to say how many they will see altogether when that row is revealed. Repeat with pairs of counters set out along a number line. Encourage the child to step over the pairs with their finger as they count up in twos. How many twos? What calculation have we just done?	Encourage counting aloud in twos and tens using images and resources such as number lines, counters and arrays. Make the connection between the multiplication facts written as a table: $2 \times 1 = 2$ $2 \times 2 = 4$, etc.
Multiply two or ten by a single-digit number, by counting up in twos and tens from zero. **2 Y2**	Does not link counting up in equal steps to the operation of multiplication; does not use the vocabulary associated with multiplication. ***2 Y2* ×/÷**	How can you quickly work out two multiplied by seven? What steps could you count up in to help you? How many steps do we need?	Using seven paper strips each showing two dots, make an array seven rows by two columns. With the child, arrange the strips into a 'number line' of seven pairs of dots. Count up in twos to fourteen. Count the steps and record $2 \times 7 = 14$. Rearrange the strips back into the seven by two array and emphasise the link to two multiplied by seven equals fourteen. Add and remove strips and repeat.	Display the number statement $10 \times 6 = ?$ Ask the child to use paper strips with ten dots on each, to form an array that represents ten multiplied by six. How many dots are there? What if you were to arrange the strips in a number line? How many dots would there be? How do you know? Ask the child to choose a different array.

Tracking children's learning through the *NNS Framework for teaching mathematics* (multiplication and division)

Associated knowledge skills	Errors and and misconceptions	Questions to identify errors and misconceptions	Teaching to address the errors and misconceptions	Next steps in moving towards the key objective
Interpret an array as a repeated addition and as a multiplication, and recognise how the array can be described as, for example, 3 + 3 + 3 + 3, 3 × 4, or as 4 + 4 + 4, 4 × 3. **3 Y2**	Does not focus on 'rows of' or 'columns of', but only sees an array as a collection of ones. ***3 Y2* ×/÷**	Display six rows of two. How many rows are there? How many dots are there in each? How can you describe the array? Turn the array through 90°. How has the array changed? Can you describe the array to me now? What's the same and what's different?	Display the first row of a four-by-five array. Here are five dots. There are four rows, which are exactly the same. Make the array with these counters. Tell me about your array. What numbers could you add together to find the total? What numbers could you multiply?	Reveal the first row and first column of a four-by-five array of dots.  Describe this array to me. Write a number sentence to match the array. Make up a number sentence that matches another array and draw the array.
Recognise that doubling and multiplying by two are the same, and use known multiplication facts and partitioning to double numbers to fifteen. **4a Y2** **4b Y2**	Has difficulty relating multiplying by two to known facts about doubles; records double four as 4 + 4. ***4a Y2* ×/÷** Does not use partitioning to find double twelve or double thirty-five. ***4b Y2* ×/÷**	What is six multiplied by two? How did you work it out? What is double six? What did you notice? What is the answer to double ten? …double four? How can we use these answers to find double fourteen? How can we work out double thirty five?	Use a bead string to make a set of five beads. Ask what double this set of beads would be. Make another set of five beads. Use the vocabulary double five beads, five beads twice, five beads multiplied by two. Agree that there are ten beads in each case and record: Double 5 = 10 and 5 × 2 = 10. Use the beads to double ten. Make a set of fifteen beads. Separate into ten and five and double the ten and five. Combine to make thirty. Record: Double 15 = Double 10 + Double 5 = 20 + 10 = 30. Repeat for other two-digit numbers.	Illustrate the following with Base apparatus and ask the child to continue these patterns, using partitioning as appropriate. 2 + 2 = 4 3 + 3 = 6 4 + 4 = 8, etc. 10 + 10 = 20 11 + 11 = 22 12 + 12 = 24 13 + 13 = 26, etc. 20 + 20 = 40 25 + 25 = 50 30 + 30 = 60 35 + 35 = 70, etc. Use appropriate vocabulary as you record these patterns as multiplications, for example: 3 × 2 = 6 13 × 2 = 26 25 × 2 = 50

Tracking children’s learning through the *NNS Framework for teaching mathematics* (multiplication and division)

Associated knowledge skills	Errors and and misconceptions	Questions to identify errors and misconceptions	Teaching to address the errors and misconceptions	Next steps in moving towards the key objective
Recognises that when finding the double of a number, half the answer is the original number. Uses the inverse of double to find halves of small even numbers to ten, using known facts. **5 Y2**	Does not use knowledge of doubles to find half of a number; for example, continues to find half by sharing, using a ‘one for you’ approach and cannot apply knowledge of doubles. ***5 Y2 ×/÷***	What do you think half of ten is? How do you know? Can you think of a number that’s easy to halve? Why do you think it’s easy to halve your number?	Tell the child you have three counters in your left hand and three in your right hand. How many counters have I altogether? Give the child eight counters. You have double the counters I have in my hand, how many have I got? Show four numbers. What is double four? What is half eight? etc.	I think of a number and double it and my answer is eight. What number was I thinking of?
Share a given number of objects out equally, recognise the relationship between sharing equally and division and use the vocabulary of division to describe the process, for example ‘divide by’, ‘share equally’. **6 Y2**	Is not systematic when sharing into equal groups, using a ‘one for you’ approach; does not use the language of division to describe the process. ***6 Y2 ×/÷***	Here are twelve counters, share them out equally into these three boxes. How many counters are there in each box? Here are six coins, can you divide them between three purses? How many will be in each purse? Altogether we had eighteen counters and there are six in each box, can you describe what we have done?	Share a set of picture cards equally between a number of players. Describe the process using the correct language of division, for example ‘twenty cards divided between four players gives each player five cards’.	At the camp there are six tents. The children wonder how many will be sleeping in each tent. They know that there are eighteen children in the group, and there has to be the same number in each tent. How many sleep in each tent?
Begin to understand division as repeated subtraction or grouping. **7 Y2**	Does not understand that ‘sets of’ or ‘groups of’ need to be subtracted to solve the problem. ***7 Y2 ×/÷***	There are fourteen children and the children are asked to work in pairs (twos). How many pairs are there?	Show the child twelve cubes. Ask them to take two cubes and make a tower. Repeat to make another tower and again to make a third tower. Model the language at each stage, for example ‘four cubes makes two towers’. Then ask the child how many towers of two they can make using twelve cubes, showing the array of two rows by six columns so that the they can relate the problem to a visual image.	Invite the child to use twelve cubes to make towers of three and ask them to explain how many towers can be made. Repeat with the towers of four and six. Ask the child to chose an even number of cubes and predict how many towers of two they would be able to make. Get them to check their answer against their prediction and reflect on what they notice.

Tracking children's learning through the *NNS Framework for teaching mathematics* (multiplication and division)

Reception key objective Use developing mathematical ideas and methods to solve practical problems (NNS *Framework for teaching mathematics*,Supplement of Examples, section 4, page 20)

Associated knowledge and skills	Errors and misconceptions	Questions to identify errors and misconceptions	Teaching to address the errors and misconceptions	Next steps in moving towards the key objective
Count in twos and talk about how many pairs. **1 YR**	Confuses numbers when counting in twos; has difficulty understanding a pair consists of two objects. ***1 YR* ×/÷**	Can you match this sock/glove to make a pair? Can you find some more pairs? How many pairs have you made?	Find matching pairs from a collection of pictures or objects, introducing the vocabulary within the context of the activity and encouraging the child to use the correct vocabulary.	Play snap/pelmanism and count in twos to see how many pairs of cards you have made. How many pairs of eyes on these three faces? If you draw four faces how many pairs of eyes would you need? How many eyes? What if you draw…? Extend to non-matching pairs, for example pairs of children.
Say how many altogether in a double; recognise that finding a double means forming a pair or adding a number to itself. **2 YR**	Has difficulty with identifying doubles and adding a small number to itself, for example 2 + 2, to make twice as many. ***2 YR* ×/÷**	Within role play, engage with child and ask questions, for example: If there are two wheels on this scooter, how many wheels on two scooters? How many gloves make a pair?	Show one animal/object and add a second one (double it). Stress vocabulary and encourage child to explain what you are doing.	Extend to a range of doubles contexts, for example: Put four cows in the field, now put double that number in the next field. How many altogether? What is double four? What if you put five cows in the field this time and double that number in the next field…?

Tracking children's learning through the *NNS Framework for teaching mathematics* (multiplication and division)

Associated knowledge skills	**Errors and and misconceptions**	**Questions to identify errors and misconceptions**	**Teaching to address the errors and misconceptions**	**Next steps in moving towards the key objective**
Solve simple problems (where there are no remainders) that involve counting the numbers of objects by organising them into groups of equal size, including more than two groups; talk about the number of groups and objects, and record their answers in their own way. **3 YR**	Makes unequal groups and is unable to compare the groups. ***3 YR* ×/÷**	Can you put the same number of spots on each wing of the ladybird? Can you share out these biscuits fairly so there are the same number of biscuits on each plate?	Practical activities set in context; for example planting bulbs, giving out stickers, sharing biscuits. Relate to stories such as *The Doorbell Rang*, by Pat Hutchins (Mulberry Books, 1986, ISBN 0 6880 9234 9)	Extend to considering groups and objects. How many plates for three dogs? How many dog biscuits do we need so that they can have three each? Can we put some marks/numbers on each plate to help us remember? What if we had twelve biscuits – how many for each dog?
Share objects out fairly into two or more groups, talk about whether there will be any left over. **4 YR**	When sharing, can sometimes make equal groups, but has no strategies to deal with any left over. ***4 YR* ×/÷**	Can you share these sweets equally between you and a friend? Will there be any left over? Can you share these marker pens between the group? Will there be any left over? What will you do with any left over ones?	Practical sharing activities related to different contexts; for example sharing classroom resources, making groups for PE.	Extend by linking to making simple predictions. Can you share these seven cubes equally into two groups? Why not? Can you choose a number of cubes that will share equally with none left over?
Count in tens forwards and backwards from a tens number (i.e. multiple of ten), and identify the tens number before and after a given tens number. **5 YR**	Has difficulty with counting reliably in tens from a multiple of ten. ***5 YR* ×/÷**	Can you say all the numbers marked on this number line? For example, ten, twenty… Can you put the cards in the correct order so we can count in tens to one hundred?	Provide visual resources such as a 100-square, a tens number track, large numeral cards from 10 to 100 to hold up to demonstrate counting forwards/backwards in tens from different starting numbers. Provide clues to identify particular numbers, for example: Can you stand on the tens number that comes just after 60?	Extend to mental imagery using a 100-square, for example 'Look carefully at 10. Close your eyes (or look at the ceiling). Can you still see the10? Now look at the 10 and the 20. Close your eyes (or look at the ceiling). Can you see the 10 and 20? What number comes next? etc.
Solve problems that involve finding halves. **6 YR**	When halving, makes two unequal groups or splits a single object unequally. ***6 YR* ×/÷**	How many sheep in the field? Can you put half of them in the pen? How many are in the pen? How many are in the field?	Practical activities with different numbers and types of objects to experience halving. Include activities involving measures; for example capacity, mass, money.	Extend to activities that involve the child in finding contexts for halving.

Appendix 3: Handouts for staff meeting

Handout 1: *Structure of Wave 3 mathematics materials*

Handout 2: *Mathematical themes and assessment approaches in the Wave 3 materials*

Handout 3: *Assessment and planning process*

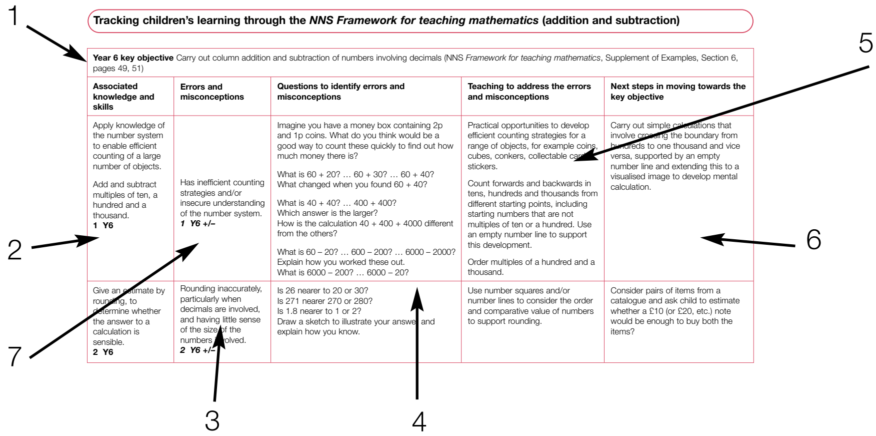
Handout 1

Structure of Wave 3 mathematics materials

In order to exemplify progression in calculation, Reception, Year 2, Year 4 and Year 6 have been chosen as representative milestones. Under each year group heading, associated knowledge and skills that contribute to understanding of the year group key objective are listed. (See first column of tracking chart.)

The whole primary age range is represented in the progression in the chart. The year group labels provide a convenient link to the National Numeracy Strategy *Framework for teaching mathematics* progression in number and calculation. As they use the chart, teachers will need to 'track back' to find the error or misconception appropriate for the child, ***irrespective of the year group to which it is attributed in the progression***.

Tracking charts



Tracking children's learning through the *NNS Framework for teaching mathematics* (addition and subtraction)

Year 6 key objective Carry out column addition and subtraction of numbers involving decimals (NNS *Framework for teaching mathematics*, Supplement of Examples, Section 6, pages 49, 51)

Associated knowledge and skills	Errors and misconceptions	Questions to identify errors and misconceptions	Teaching to address the errors and misconceptions	Next steps in moving towards the key objective
Apply knowledge of the number system to enable efficient counting of a large number of objects. Add and subtract multiples of ten, a hundred and a thousand. **1 Y6**	Has inefficient counting strategies and/or insecure understanding of the number system. ***1 Y6 +/–***	Imagine you have a money box containing 2p and 1p coins. What do you think would be a good way to count these quickly to find out how much money there is? What is 60 + 20? … 60 + 30? … 60 + 40? What changed when you found 60 + 40? What is 40 + 40? … 400 + 400? Which answer is the larger? How is the calculation 40 + 400 + 4000 different from the others? What is 60 – 20? … 600 – 200? … 6000 – 2000? Explain how you worked these out. What is 6000 – 200? … 6000 – 20?	Practical opportunities to develop efficient counting strategies for a range of objects, for example coins, cubes, conkers, collectable cards, stickers. Count forwards and backwards in tens, hundreds and thousands from different starting points, including starting numbers that are not multiples of ten or a hundred. Use an empty number line to support this development. Order multiples of a hundred and a thousand.	Carry out simple calculations that involve crossing the boundary from hundreds to one thousand and vice versa, supported by an empty number line and extending this to a visualised image to develop mental calculation.
Give an estimate by rounding, to determine whether the answer to a calculation is sensible. **2 Y6**	Rounding inaccurately, particularly when decimals are involved, and having little sense of the size of the numbers involved. ***2 Y6 +/–***	Is 26 nearer to 20 or 30? Is 271 nearer 270 or 280? Is 1.8 nearer to 1 or 2? Draw a sketch to illustrate your answer and explain how you know.	Use number squares and/or number lines to consider the order and comparative value of numbers to support rounding.	Consider pairs of items from a catalogue and ask child to estimate whether a £10 (or £20, etc.) note would be enough to buy both the items?

1 Key objective.

2 This column lists associated knowledge and skills that contribute to understanding of the key objective.

3 Common errors and misconceptions linked to specific knowledge and skills are listed to support diagnosis of children's difficulties.

4 Questions in this column can be used to help the teacher decide where the child's difficulties lie.

5 Examples of the types of teaching activity in the A4 booklets (see below).

6 This column provides ideas to develop when the child has improved their understanding of the identified difficulty. The teacher can make use of these ideas to consolidate understanding and extend thinking.

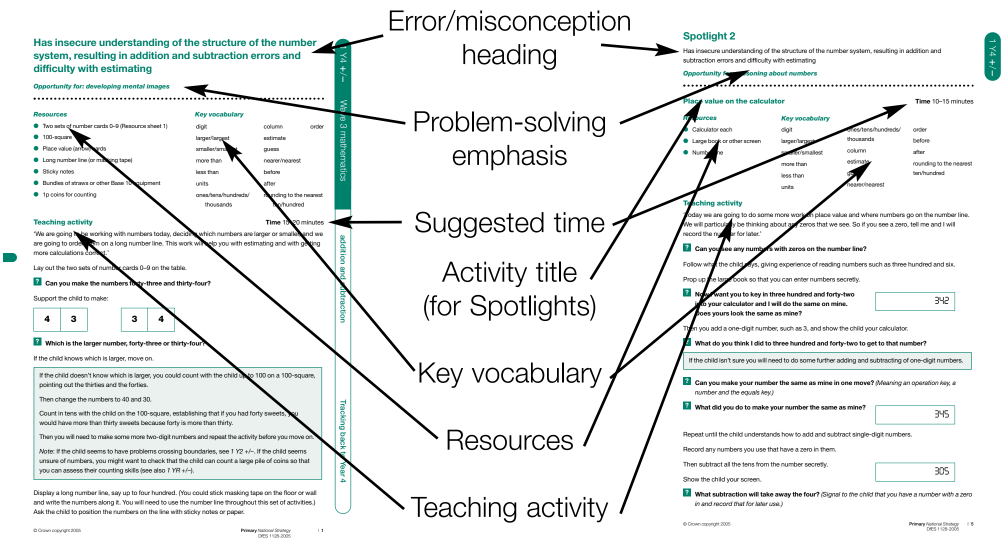
7 Code referencing to an A4 teaching unit.

A4 booklets – teaching units

The structure of each booklet is as follows:

- focus error/misconception;
- opening teaching activity addressing error/misconception;
- a number of Spotlights (short focused teaching activities from which to select);
- final Spotlight, which includes assessment opportunities, often encompassed in a game, key vocabulary checklist, and intended learning outcomes list;

Opening teaching activity ***Spotlight***



Specific icons are used to improve access to the text:

Icons

Questions are incorporated for teachers to select from and add their own as appropriate.

Whole-class follow-on activity.

Symbol reminding of the necessity to estimate, calculate, then check.

This variation of the game is harder.

This variation of the game is easier.

12 x 2 = 24

Text within this symbol indicates an opportunity for recording.

Text within a shaded box indicates alternative approaches for a child who is having difficulty with the activity.

Additional game at the end of some teaching units.

Handout 2

Mathematical themes and assessment approaches in the Wave 3 materials

The following are fundamental to the approach in these Primary National Strategy materials.

- Using and applying mathematics has been integrated. Often there are several opportunities for problem solving within one activity, but in each, one particular opportunity has been highlighted. Aspects such as the following are incorporated:
  - encouraging children to discuss and explain in order to support development of their mathematical reasoning;
  - opportunities for children to make choices are woven into the activities, for example selecting numbers and devising calculations;
  - encouraging children's own recording to communicate mathematical thinking, focusing on efficiency;
  - opportunities for evaluating the efficiency of methods of calculation.
- Development is emphasised and key vocabulary is listed in each activity. It is important for adults to use correct mathematical language. To facilitate this, examples are given in words, for example, 725 × 3 is accompanied by what the adult could say to the child: 'Seven hundred and twenty-five *multiplied by* 3'.
- There is a focus on progression in counting from the earliest stages through to Year 6 to support the development of secure counting skills.
- Throughout the materials, there is emphasis on the process of estimating first, then calculating and then checking. This is denoted by the following icon: 
- Decimals are addressed within meaningful contexts, for example via displays on a calculator and as a part of measure.
- Structured equipment and everyday materials are used to model mathematical concepts, supporting children's mathematical thinking and development of mental imagery. Some links to ICT resources such as the Primary National Strategy Interactive Teaching Programs (ITPs) are included.
- A wide range of resources is used in the teaching sessions. Teachers' selection of these to suit the needs of their children is an important part of adapting the materials.

The materials reflect best practice in assessment for learning as a key tool for raising achievement through:

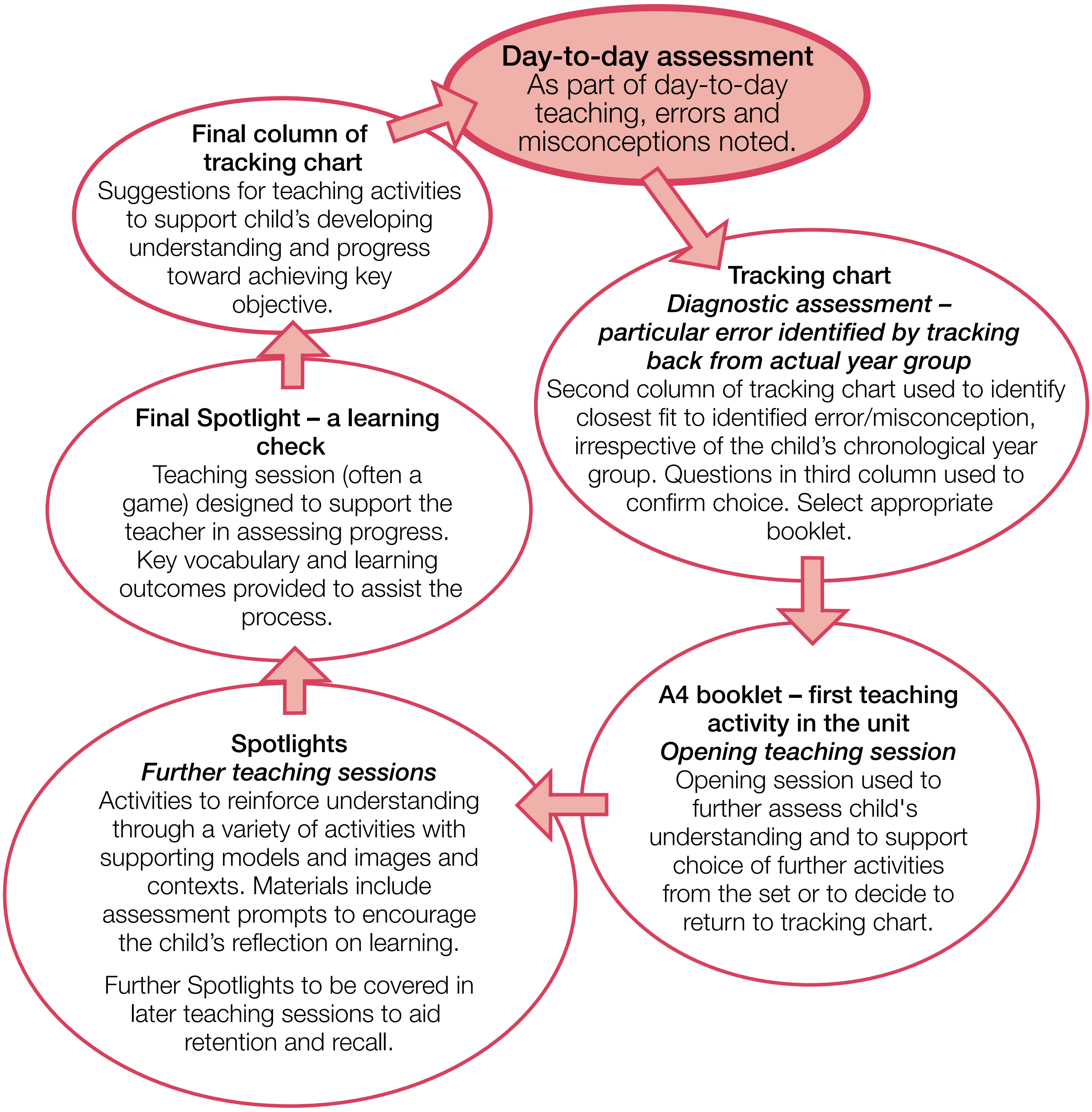
- use of questions to elicit information about children's understanding;
- sharing the purpose of the activity with the learners;
- encouraging children's reflection on their learning and identification for themselves of possible next steps.

Handout 3

Assessment and planning process

Day-to-day assessment is the starting point. Assessment opportunities are embedded in all the teaching materials.

The following flow chart describes the process assumed in the design of these Primary National Strategy materials.



Appendix 4: Further references

What works for children with mathematical difficulties? (DfES research report 554). This is available from DfES publications. (tel: 0845 60 222 60) or can be downloaded from the website at www.dfes.gov.uk/research.

Using models and images to support mathematics teaching and learning in Years 1 to 3 (DfES 0508-2003 GCDI)

Including all children in the literacy hour and daily mathematics lesson (DfES 0465/2002)

Guidance to support pupils with specific needs in the daily mathematics lesson (DfES 0545/2001)

Framework for teaching mathematics from Reception to Year 6

Primary National Strategy website: www.standards.dfes.gov.uk/primary

Appendix 5: Resources

Equipment

The lists below show all the mathematical equipment and everyday materials required in the teaching activities.

There is an assumption that regular classroom stationery, such as paper (including squared paper), felt tip pens, pencils, labels, sticky notes and whiteboards with dry wipe pens will be available.

Mathematical equipment	**Everyday materials**
Cubes	Small counting items, cars, animals, etc
Counters	Food items, such as apples, tangerines, biscuits
Beads	Hoops, balls, beanbags
Money, including notes	Small and large bricks
Number lines to 30/100	Cloth
Wipe-clean number line	Jars, boxes, pots, jugs, beakers
Floor or playground number line	Tins and tin tray
Number track	Draught pieces
Bead string	Game board, such as Ludo
Counting stick	Magnet and paper clips
Small 100-square	String, ribbon
Wipe-clean 100-square	Paper plates
Large 100-square with removable numbers	Cups, plates spoons
Dominoes	Tiddlywinks game
Large foam number tiles	‘Eggs’ (ping pong balls) and egg boxes
Dice	Coat hanger and pegs
Calculator	Cloth bags to hold cards
Place value (arrow) cards	Purses
Wipe-clean place value board	Paper bags
Abacus	Small character toys, for example astronauts
Bundles of straws	Fish shapes
Base 10 equipment	Framed picture
Cuisenaire or other rods	Pairs of socks/gloves
One/two-minute timer	Lolly sticks
Plastic tocker timers	Marbles
Sand timers	Sand tray
Stopwatch	Mirror
Large digital clock	Finger puppets
Teaching clock (analogue)	Construction bricks with pairs of prongs
Metre rulers	Items for rewards

Mathematical equipment *cont.*	
Measuring tape	
Weights	
Balance scales	
2-D shapes	
3-D shapes	
Straight-sided clear containers/ measuring cylinders	
Plastic or other fraction pieces	
Sorting circles	

Suggested Interactive Teaching Programs (ITPs)

Interactive Teaching Programs for teaching calculations (downloadable from the CD-ROM, and the latest versions from the Primary National Strategy website www.standards.dfes.gov.uk/primary).

The list can be added to as new programs become available.

Addition, subtraction and place value	**Multiplication and division**
Difference Uses a number line to find difference	*Number grid* Multiples shown on a 100-square
Number line Addition and difference on a number line	*Grouping* Division on a number line
Place value Hundreds, tens and units on place value cards	*Multiplication grid* Grid method of multiplying
Counting on and back Simulates a 100-bead string	*Division grid* Long division
Number facts Addition and subtraction number sentences	*Remainders after division* Gives remainder as a number or a fraction
Ordering numbers Relating counters to a number line	*Multiplication facts* Multiplication as repeated addition
Twenty cards Sequences	*Moving digits* Multiplying and dividing by 10 and 100
	Number dials Multiples